Gnomon is an Island
Whorled Polygons
Cye H. Waldman

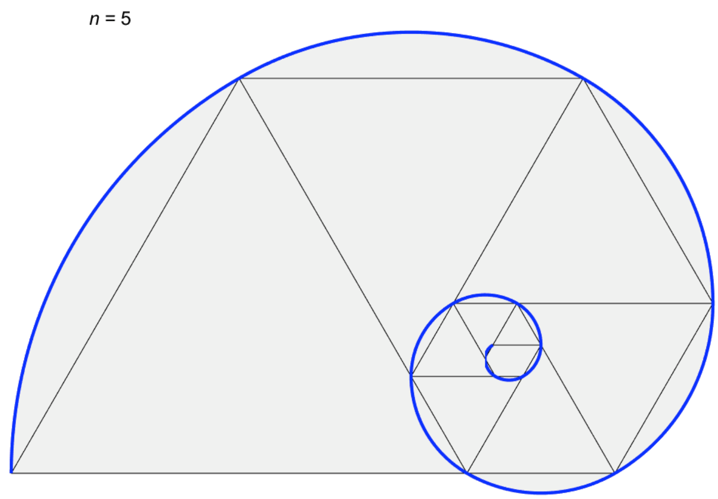

Figure 1: The plastic pentagon gnomon is an example of a whorled polygon.

Introduction

We are familiar with the concept of gnomonic tiling in which a figure added to another reproduces the shape of the original. The process can repeated over and over to form a tessellated mosaic that covers the plane. The initial tile is called the *seed* and the added piece is called the *gnomon*. The gnomon generally increases in size geometrically, i.e., by a power law. Figures thus created are called whorled figures, as the gnomons are usually added in a circular fashion about the seed. See, for example, Gazalé [1] and Waldman [2].

There are magnificent mosaics, or tilings, if you prefer, that continue to amaze us. These are the whorled plastic pentagon, with its equilateral triangle gnomon, and the whorled golden rectangle, with its square gnomon, shown in Figure 1 and Figure 2, respectively. The former has a growth rate of p, the plastic number, and the later has a growth rate of φ, the golden ratio.

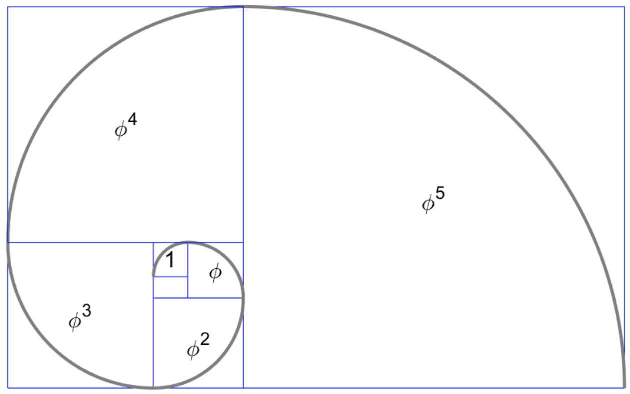

Figure 2: Golden rectangle and its square gnomon with a growth rate of the golden ratio.

Insofar as we have developed a program for creating pseudospirals and their attendant triangles (or squares), we wondered if we could find additional mosaics that covered the plane. By the way, it's fairly well established that the whorled figures in Figure 1 and Figure 2 are the only ones whose gnomons are *regular* polygons, by which we mean, of uniform sides and internal angles.

Lo and behold, we found by experimentation the pseudospirals with a sequence q^n and rotation angle $\pi/5$ produce a whorled octagon with an isosceles triangle gnomon, the growth rate q was determined empirically. The result is shown in Figure 3. Interestingly, the gnomon is the golden triangle.

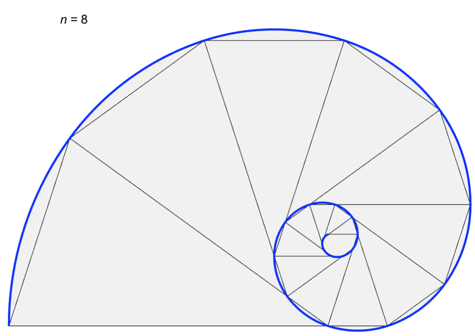

Figure 3: An octagon and its golden triangle gnomon.

This set the stage for us to (a) determine the growth rate analytically, and (b) see if the result could be generalized for other growth rate and turn angles.

Technical

Referring to Figure 4 for the nomenclature, let the 7-gon represent a *k*-gon whose sides and gnomon are to be determined, along with the gnomon growth rate. The vertex angle of the gnomon is denoted by θ and the first triangle has unit sides and a base of $2\sin\frac{\theta}{2}$.

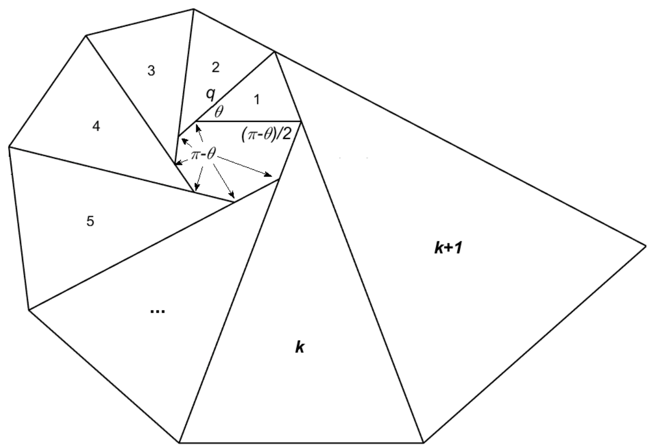

Figure 4: A generic *k*-gon with a triangle gnomon.

Now, we know that the sum of the internal angles of an arbitrary *k*-gon is equal to $(k-2)\pi$, ergo

$$(k-1)(\pi-\theta)+\tfrac{1}{2}(\pi-\theta)=(k-2)\pi$$
$$\theta=\frac{3\pi}{2k-1} \tag{1}$$

We can also reckon from 1st, k^{th}, and $(k+1)^{th}$ triangles that

$$q^{n+k}-q^{n+k-1}=q^{n}\cdot 2\sin\tfrac{\theta}{2} \tag{2}$$

Thus, the growth rate can be determine iteratively from

$$q-q^{k-1}=2\sin\tfrac{\theta}{2} \tag{3}$$

Moreover, we can work backwards and determine the sides of the polygon. Starting with the unit edge and going anticlockwise we find that the sides of the polygon grow as $q^{n}(q-1)$ for $n=0:k-1$.

There are both pseudospirals and logarithmic spirals associated with these mosaics; the latter are possible only because of the geometric growth and have a flair coefficient of $\ln q/\theta$. The pseudospirals are defined the sequence q^n and rotation angle θ.

The associated animation at the Website [3] show a spectrum of mosaics and spirals for *k* = 3 to 100. These include the plastic pentagon (*k* = 5), but not the golden rectangle whorl, for it is not triangle based. However, for *k* = 3 we have another well-known mosaic, specifically the golden triangle whorl with the golden gnomon (yes, that's a real term) and growth rate φ, the golden ratio. This is shown in Figure 5. Figure 6 shows an additional example, here, for *k* = 13.

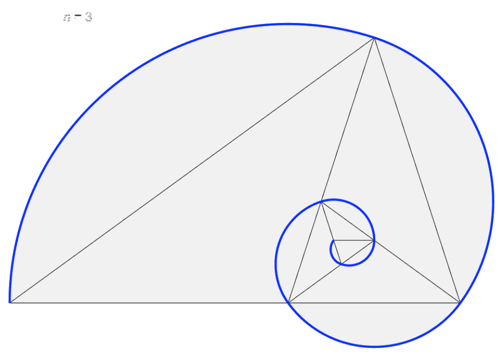

Figure 5: Golden triangle with golden gnomon.

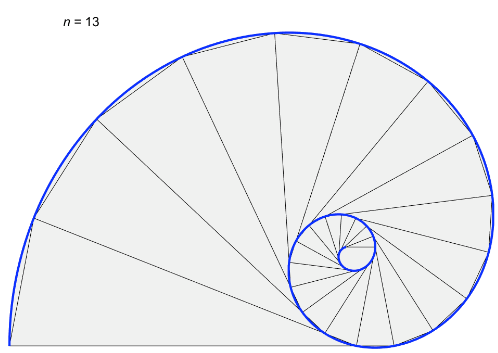

Figure 6: Example arbitrary *k*-gon with triangle gnomon (*k* = 13).

Notice that the spirals touch every vertex in the mosaic, including the seed polygon. However, the seed is left uncovered in the animation for clarity. In addition, note that the logarithmic and pseudospirals are indistinguishable at the scale of the figures and only the pseudospirals are shown. You can learn more about pseudospirals from Waldman and Gray [4].

Summary and Conclusions

We have generalized the golden triangle and plastic pentagon whorls for an arbitrary *k*-gon. It does not appear than further generalizations are possible.

References

[1] M.J. Gazalé, *Gnomon: From Pharaohs to Fractals*. Princeton University Press (1999).
[2] Waldman, C.H., Gnomons Land (2016).
http://curvebank.calstatela.edu/waldman16/waldman16.htm.
[3] Waldman, C.H., Gnomon is an Island (2016).
http://curvebank.calstatela.edu/waldman?/waldman?.htm.
[4] Waldman, C.H. and Gray, S.B, Fibonacci, Padovan, & Other Pseudospirals, submitted for publication (2016).
See also: http://curvebank.calstatela.edu/waldman6/waldman6.htm.