Tricurve Basics

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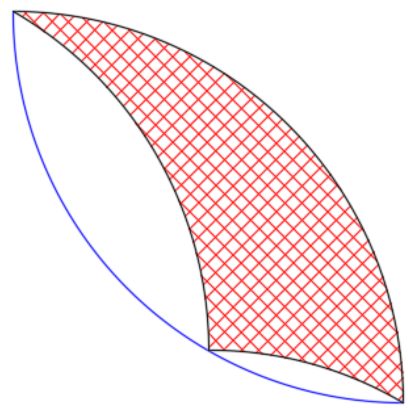


Figure 1 -- Tricurve

Introduction

When we think of substituting arcs for the straight sides of a triangle, we usually think of circular triangles based on an equilateral triangle, or else an arbelos.

In this paper we present a type of circular triangle as shown in Figure 1 which not only tesselates but has other unique properties.

Technical

The tricurve can be constructed as shown in Figure 2. In this example we start with a simple 90-degree arc. The arc is then mirrored about the line between its endpoints to make a lens shape, and the new arc is divided into the two parts such as 30 and 60 degrees. Each of these smaller arcs is in turn mirrored about a line between its endpoints, forming the two concave portions of the tricurve.

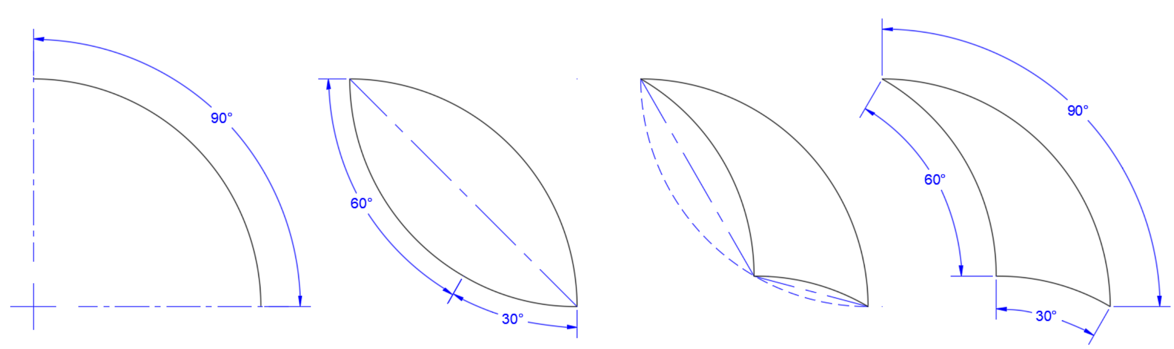


Figure 2--Construction of a 30-60-90 Tricurve

The large arc can be any angle up to 180 degrees, with the two smaller arcs in any proportion.

Figure 3 (a) shows the relationship between angle and arcs for the tricurve in Figure 1. The corner angles opposite the 30- and 60-degree arcs are 30 and 60 degrees, respectively. Each smaller arc always matches the corner angle opposite. The corner angle opposite the large arc, measured between the tangents at the arc ends, is the supplement of the arc of 90 degrees, which in this is also 90 degrees. The large arc is always opposite its supplementary corner angle. Figure 3 (b) show these relationships for a 30-90-120 tricurve.

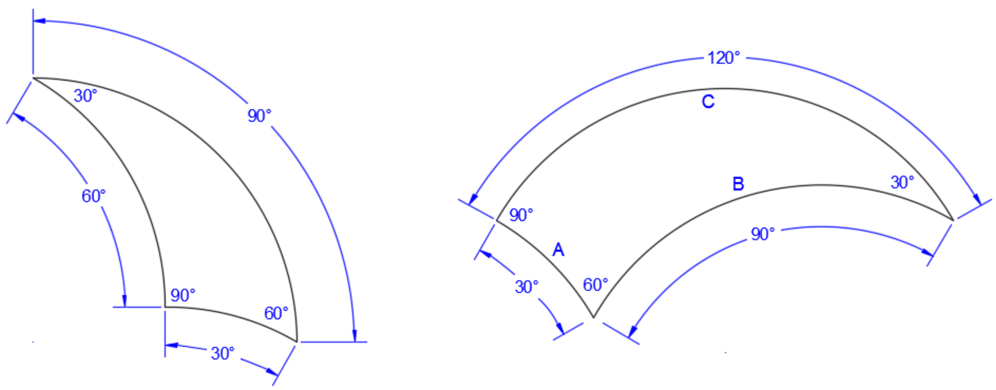


Figure 3 -- Examples of Angle-Arc Relationships

As in a normal triangle, the tricurve interior angles sum to 180 degrees. The convex and concave arc lengths are equal. The tricurve shape does not obey the Law of Sines, so the arcs and the angles can be in agreeable proportions.

Four Implications

This unique geometry results in four main implications:

1) **Ease of tiling.** The agreeable proportions of the arcs and the angles enable interesting tiling with a single shape. Figure 4 shows ten possibilities for joining a second tile to the long arc of a first.

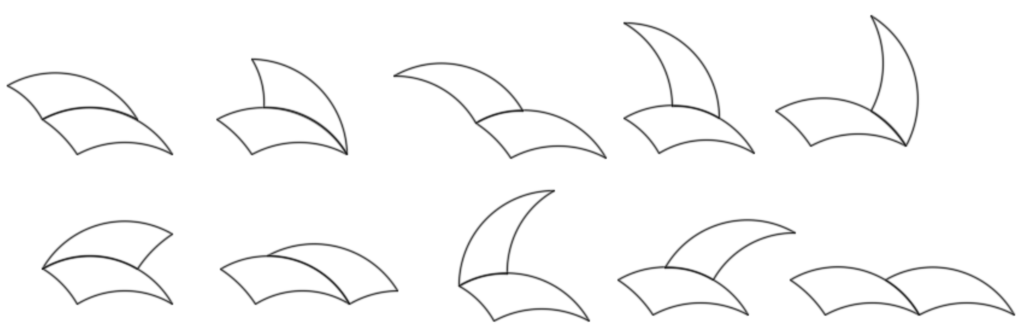


Figure 4 -- Joining Options

Figure 5 shows three ways of tiling periodically:

Figure 5 -- Periodic Tiling

Figures 6, 7 and 8 give further examples of varied tiling.

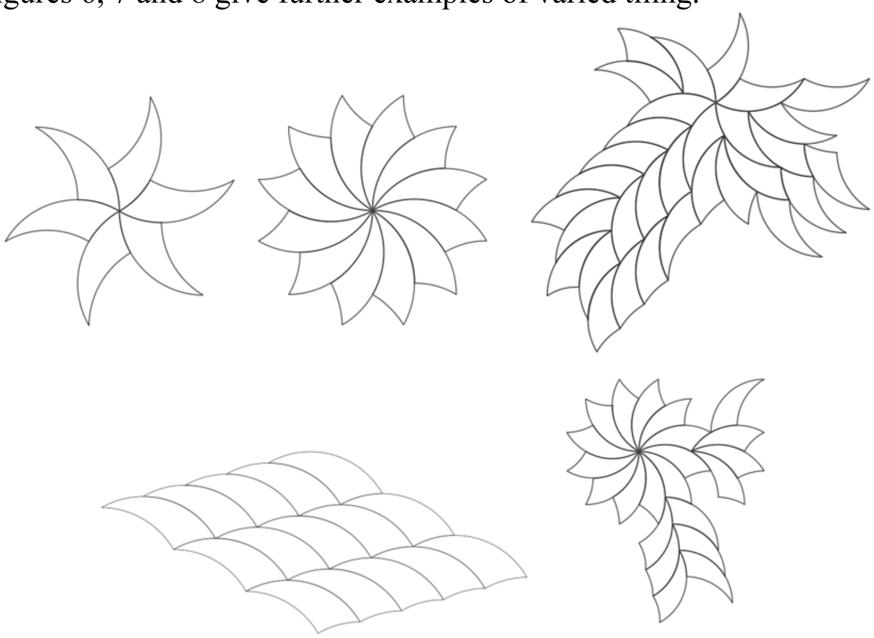


Figure 6 --Tilings with 30-60-90 tricurves

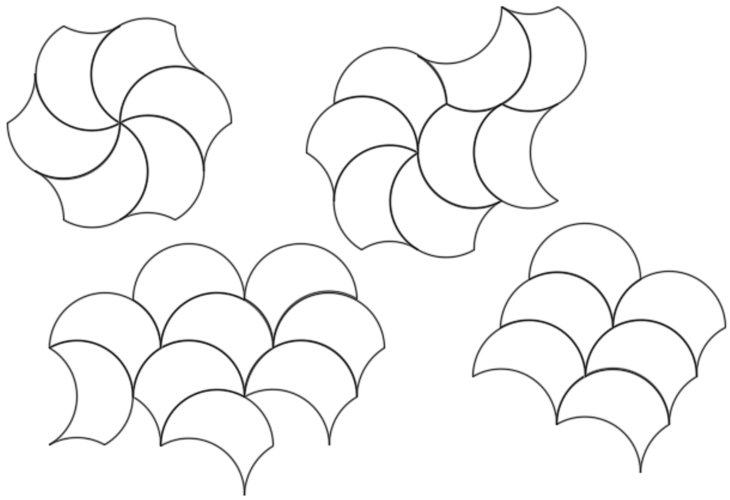


Figure 7 -- Tilings with 60-120-180 tricurves

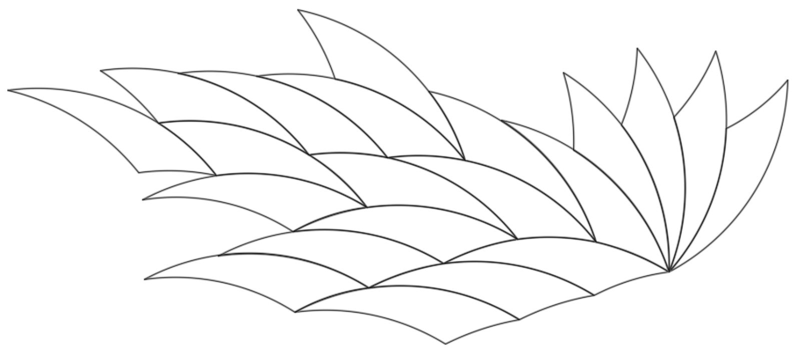


Figure 8 -- Tiling with 20-40-60 tricurves

Tiling with a single tricurve shape is similar that achieved with various faceted single tile designs such as the "bent wedge" [1], "versa-tile", and the shapes of Paul Gailiunas [2,3].

2) **Patterns of arc centers.** Figure 9 shows the three arc centers form the corners of a triangle identical to that formed by the corners of the tricurve. During tiling adjacent tricurves will share an arc center.

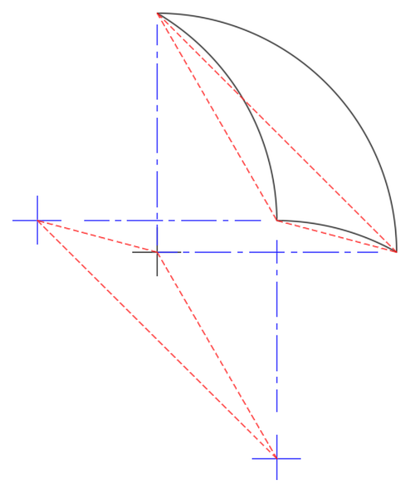


Figure 9 -- Arc Centers

3) **Unexpected Areas**. The calculation of the area of a tricurve shows that it contains portions of circles, but it is even simpler than expected. π disappears because the sector areas cancel out: the large sector has the same area as the subtracted two smaller sectors. For a unit radius, the area is the sum of the sines of the two small angles, minus the sine of the large angle. In terms of the two small angles, the area is the sum of the sines minus the sine of the sums. The calculations further simplify for certain angles and combinations of angles. See Appendix.

4) **Similar Tricurves.** While maintaining a constant radius, a type of scaled similar triangle can be made with tricurves, but not of course by simply scaling the arc lengths. The tricurve can be thought of as a sort of parallelogram of arcs, as shown in Figure 10.

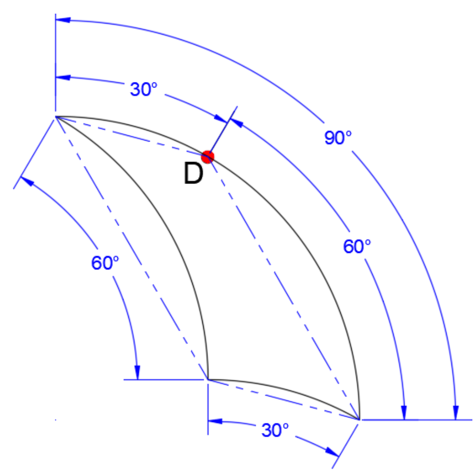


Figure 10 -- Parallelogram of Arcs

In this example the 30-60-90 tricurve can be viewed as constructed of two 30 degree arcs and two 60 degree arcs, with arc ends aligned at point D. Now if each arc is bisected and the midpoints connected with arcs, the result in the upper right quadrant is a tricurve with all arc lengths halved as shown in Figure 11.

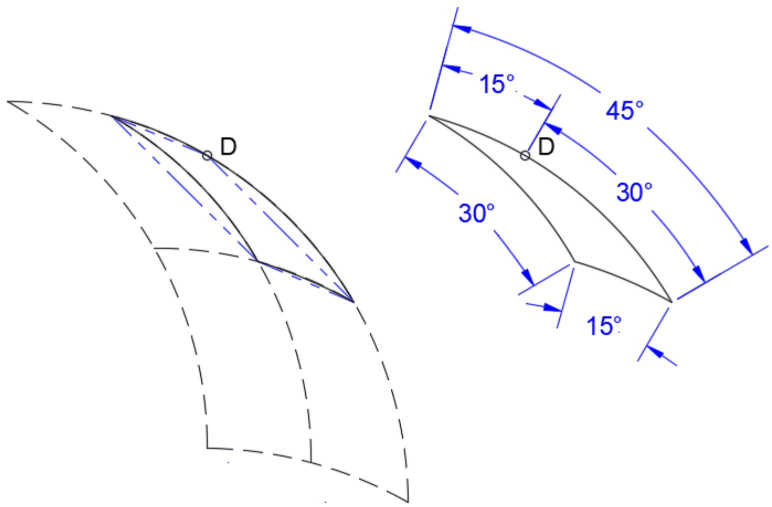


Figure 11 -- Quartered and Drawn

Note that point D stayed in place and marks the break point of the new tricurve into its curved parallelogram. Figure 12 shows that the halving of the four arcs gives us two other related tricurves, also with D as the breakpoint. This works for any scaling of the parallelogram, by proportioning the two smaller arcs, when the final long arc contains point D.

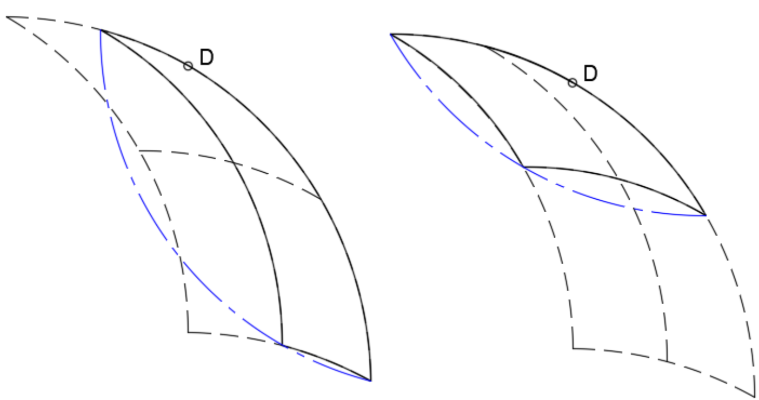


Figure 12 --Related Tricurves

Summary and Conclusions

In this paper we have presented the interesting properties and characteristics of the tricurve shape. These properties and characteristics seem to be unique among simple shapes. The tiling possibilities were only touched on briefly; these possibilities should be investigated further and hopefully will be the subject of future papers.

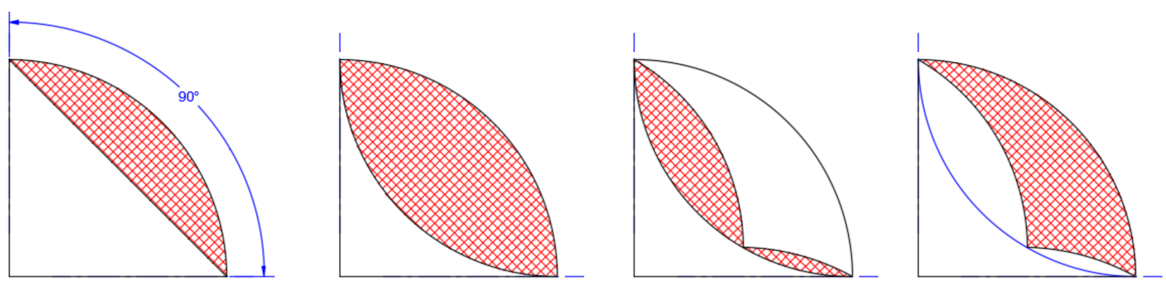
References

[1] https://www.uwgb.edu/dutchs/symmetry/radspir1.htm, by Steven Dutch, Natural and Applied Sciences, University of Wisconsin - Green Bay
[2] "Some Monohedral Tilings Derived From Regular Polygons" by Paul Gailiunas, 2008 (http://www.mi.sanu.ac.rs/vismath/galiunas2008/
[3] "Spiral Tilings", Paul Gailiunas, 2000 http://vismath4.tripod.com/gal/

Appendix

The Area of a Tricurve

The key part is the segment: the area bounded by a chord and its associated arc. This segment area is the difference between the sector area, and the triangular area bounded by the chord and the two radial lines. This is shown below left for a 90-degree angle/arc.



The segment area is then doubled, for the area bounded by a lens shape: the arc and its mirror image about its chord. The same is done for each of the two smaller arcs (30 and 60 degrees, above). The area of the tricurve is the area of the large lens (double-segment) minus the areas of the two smaller lenses (double-segments).

From any mathematical handbook we know that the areas of the segment and sector are given by

$$A_{sector} = \tfrac{1}{2} r^2 a$$

$$A_{segment} = \tfrac{1}{2} r^2 \left(a - \sin a \right).$$

For the double segment (abbreviated DS) this area is, of course, doubled:

$$A_{DS} = r^2\left(a - \sin a\right)$$

For the whole tricurve, the area is the large-angle double segment minus the two small-angle double segments. This can be expressed in terms of small angles a and b, and their total (a+b) which is the large angle.

$$\begin{aligned} A_{tricurve} &= A_{DS}(a+b) - A_{DS}a - A_{DS}b \\ &= r^2\left[(a+b) - \sin(a+b) - a + \sin a - b + \sin b\right] \\ &= r^2\left[-\sin(a+b) + \sin a + \sin b\right] \end{aligned}$$

since $A_{sector(a+b)} = A_{sector(a)} + A_{sector(b)}$.

Consider the radius as one unit length, then the area $A_{tricurve} = \sin a + \sin b - \sin(a+b)$ in square units.

Thus the sector areas cancel out: the large sector has the same area as the subtracted two smaller sectors. For a unit radius, the area is the sum of the sines of the two small angles, minus the sine of the large angle. In terms of the two small angles, the area is the sum of the sines minus the sine of the sums. This simplifies further when we use agreeable angles (say, multiples of 15 degrees) in certain combinations.

The resulting areas are shown for some cases in the table below. In Part A the large angle is 180 (with sine of zero), and the tricurve is symmetrical with the two small angles the same. Therefore, the area is twice the sine of the small angle. In Part B the large angle and one of the small angles have the same sine and so cancel out, and the area is simply the sine of the 2nd small angle.

	large angle	**1st small angle**	**2nd small angle**	Area = 2xsin:
Part A	180, with sine of zero	with same sines		
		90	90	2
		120	60	SQRT 3
		135	45	SQRT 2
		150	30	1
Part B	same sines: cancel out			Area = sine:
	165	15	150	1/2
	150	30	120	half of SQRT 3
	135	45	90	1
	120	60	60	half of SQRT 3
	105	75	30	1/2
	112.5	67.5	45	half of SQRT 2
	157.5	22.5	135	half of SQRT 2