

Brave New Whorled
Anamorphic Tiling
Cye H. Waldman
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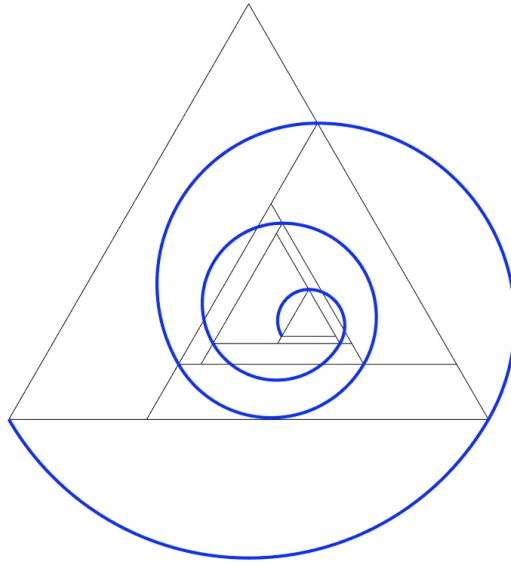


Figure 1: A generic anamorphic tiling and one of three possible spirals.

Introduction

We are familiar with the concept of gnomonic tiling in which a figure added to another reproduces the shape of the original. The process can be repeated over and over to form a tessellated mosaic that covers the plane. The initial tile is called the *seed* and the added piece is called the *gnomon*. The gnomons generally increase in size geometrically, i.e., by a power law. Figures thus created are called whorled figures, as the gnomons are usually added in a circular fashion about the seed. See, for example, Gazalé [1] and Waldman [2].

In this paper we consider a particular whorled triangle and demonstrate that it can be generalized to arbitrary, indeed, even random growth, provided that the growth is monotonic. Thus we can create anamorphic tilings for which the ‘gnomons’ are all of different shapes, albeit within the same general family. Here, we are using the term anamorphic in the optical sense of having unequal magnifications along two axes perpendicular to each other. Figure 1 shows a generic anamorphic tiling and a single spiral. In fact, we demonstrate that these tilings support three different spirals and comment on the nature of those spirals.

Technical

Figure 2 shows the layout for the calculation of the trapezoidal gnomons for an equilateral triangle. For an arbitrary sequence, say $S = \{s_1, s_2, s_2, \dots, s_n, \dots\}$, the trapezoids have a side length and height given by

$$\begin{aligned}
 l_n &= s_n - s_{n-1} \\
 h_n &= l_n \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} (s_n - s_{n-1})
 \end{aligned}
 \tag{1}$$

The only requirement here is the sequence S is monotonically increasing. However, it can be readily shown that if this is a geometric sequence, e.g., $s_n = q^n$, where $q > 1$, that the ratio s_n/h_n is constant the gnomons are self-similar, as in the classic case of gnomonic tiling.

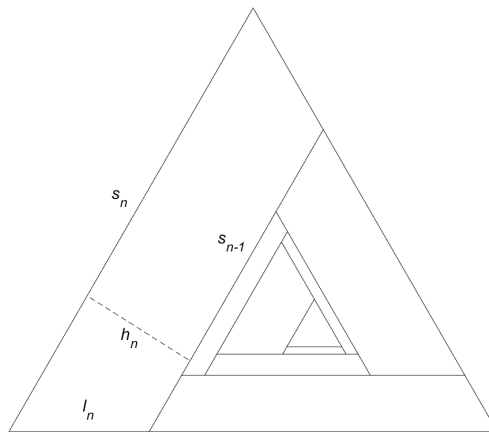


Figure 2: Nomenclature for anamorphic triangle tiling.

Figure 3 demonstrates three spirals can be shown to pass through some or all of the vertices of the triangles. By arbitrary convention, we denote the primary spiral (blue) as the one starting at the left vertex of the seed triangle and passing through two vertices of each triangle. The spiral must unroll in the same direction as the whorl. The complementary spiral (green) passes through all the remaining vertices, and likewise unfolds in the same direction as the whorl. Finally, the full spiral (red) passes through all the vertices and unfolds in the anti-whorl direction.

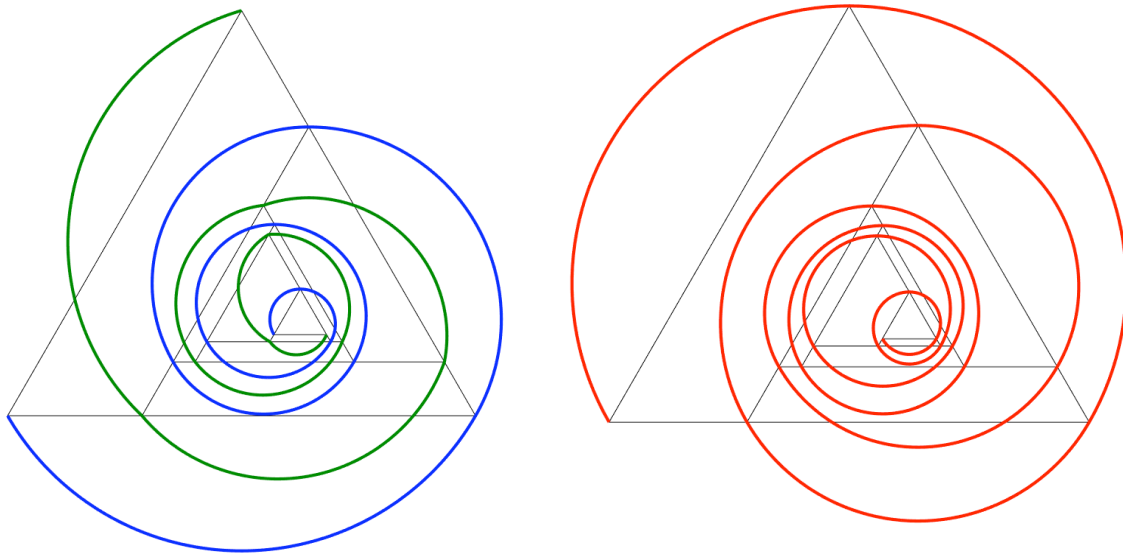


Figure 3: A generic anamorphic tiling and all three possible spirals.

What you should know about these spirals: They are all composed of circular arcs. The arcs have their origins at the centroid of the defining triangle. The defining triangles for the primary and full spirals are the gnomonic triangle themselves, and have a radius that is the side of the triangle divided by $\sqrt{3}$, i.e., $r_n = s_n/\sqrt{3}$. The defining triangle for the complementary spiral is fundamentally different as it must span two triangles at a time. The side of the triangle is, in fact, the diagonal of the trapezoid.

Referring to Figure 4, the side of the defining triangle, a is given by

$$a^2 = \left(\frac{\sqrt{3}}{2}\right)^2 (s_n - s_{n-1})^2 + \frac{1}{2}(s_n + s_{n-1})^2 = (s_n - s_{n-1})^2 + s_n s_{n-1} \quad (2)$$

And the radius is again the side divided by $\sqrt{3}$.

Now, it turns out that the primary and full spirals are regular pseudospirals, with a radius sequence of $r_n = s_n/\sqrt{3}$, and a rotation of $\Delta\theta = 2\pi/3$ and $\Delta\theta = 4\pi/3$, respectively. The complementary spiral does not conform the pseudospiral paradigm as defined by Waldman [3] and must be crafted by an *ad hoc* step-and-arc algorithm. The advantage of the pseudospiral is that there is a pro-programmed analytic solution. However, we note that when the gnomons are self-similar, the complementary spiral is also a pseudospiral.

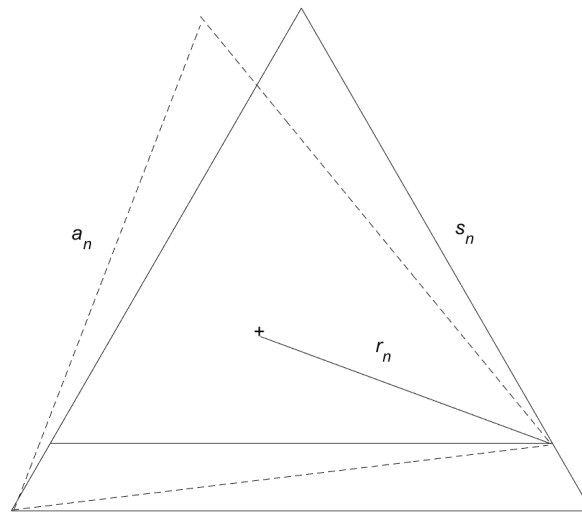


Figure 4: Geometry for the complementary spiral.

One final remark on the spirals: It is common to find comparable logarithmic spiral when there is a geometric growth. However, this is not the case here. True, the triangles grow geometrically, but the radius does not. That is because it is centered at the centroid of the triangle rather than the vertex. If the radii *were* centered at the vertices, then the spirals would be grotesquely scalloped.

Sample Results

Figure 5 shows a standard gnomonic tiling with a growth rate p^n , where p is the plastic constant. Figure 6 shows an anamorphic tiling with a growth rate based on the Padovan sequence. And finally, Figure 7 shows an anamorphic tiling for a (sorted) random sequence. In all cases we show the primary (blue) and complementary (green) spirals on the left, and the total spiral (red) on the right. Again, we remind you that in an anamorphic tiling the trapezoids are not self-similar.

The associated animations on the Web page [4] shows a sequence of gnomonic tilings; the growth ranges from 1.1 to π .

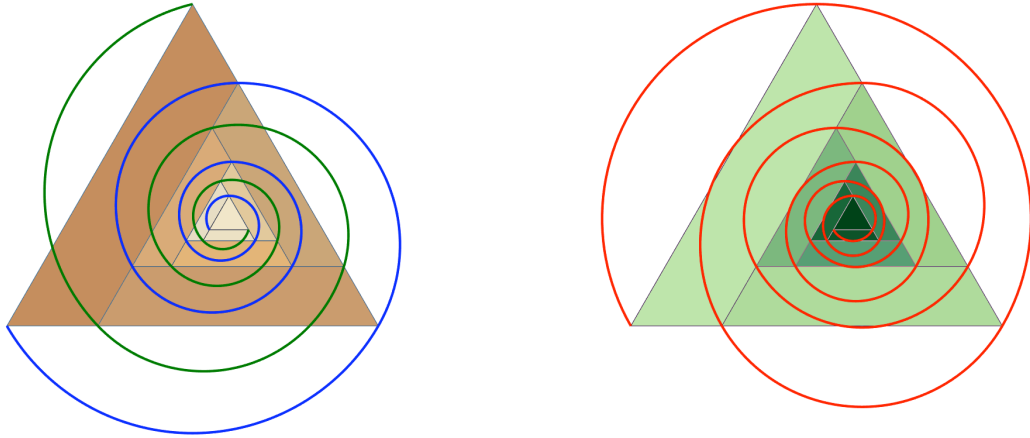


Figure 5: Gnomonic tiling with geometric growth rate, p^n .

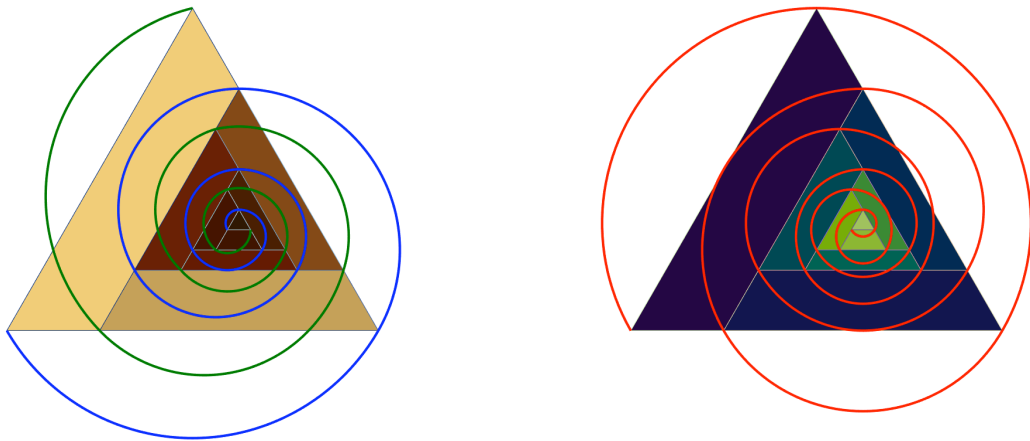


Figure 6: Anamorphic tiling and spirals with Padovan sequence growth.

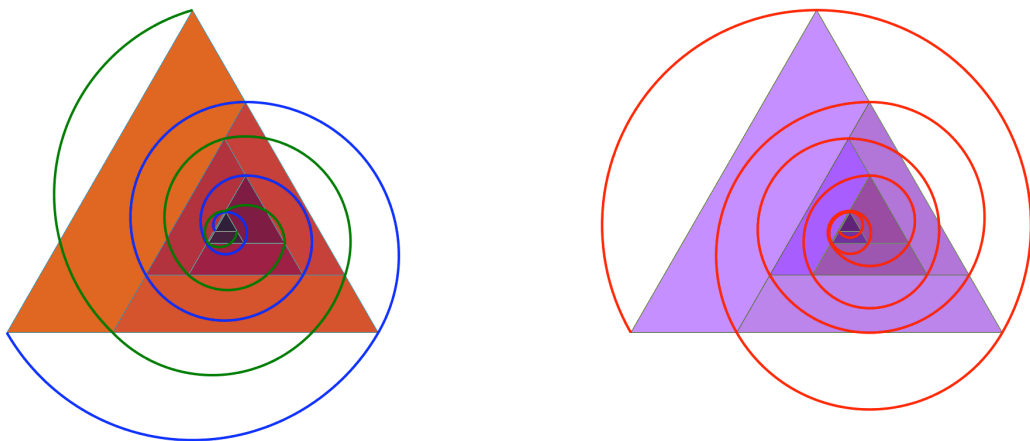


Figure 7: Anamorphic tiling and spirals with random sequence growth.

Project

It seems plausible to generalize this one step further by using an arbitrary triangle seed. In that case it makes more sense to generate larger and larger triangles and to attach them sequentially in the correct position. Figure 8 shows two examples for random configurations.

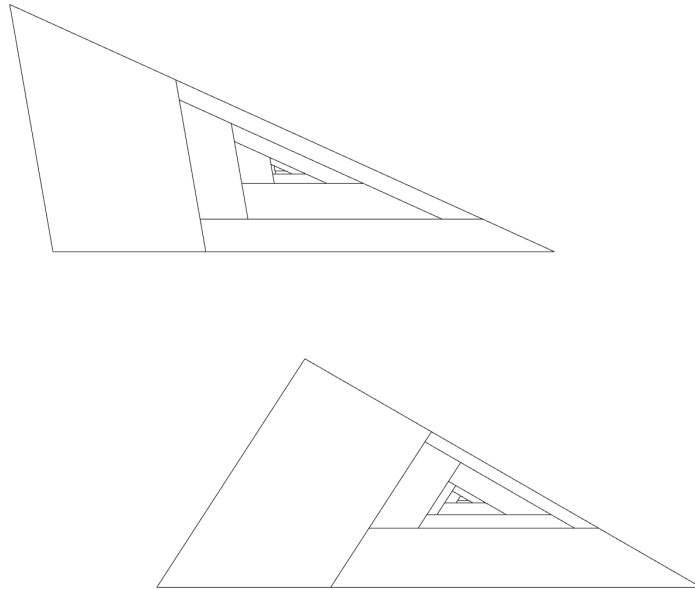


Figure 8: Example of fully random triangle tiling..

Computer program

A Matlab computer program to produce the mono- and dignomonic tilings in this paper is provided in the Appendix. It's quite simple to use. If there are no input parameters beyond the number of tiles desired, the program produces a monognomonic tiling with $\varphi_s = \varphi_r = \Phi_{crit}$. If there is one input additional parameter, the program produces a normal monognomonic tiling with $\varphi_s = \varphi_r$. If there are two more input parameters (φ_s and φ_r , respectively) the program produces a regular dignomonic tiling; however, if the second input is 'empty,' that is, input as [], then the program will set $\varphi_r = \Phi_{crit}^2 / \varphi_s$. In all cases, Gazalé's spiral and its complement are drawn over the tiling. The only requirement is that the growth rate $\Phi = \varphi_r \varphi_s > 1$.

Summary and Conclusions

We have made a modest addition to Gazalé's work on dignomonic tiling and spiral. We have demonstrated an additional spiral that complements Gazalé's in that it crosses all the other vertices. Moreover, we analytically determined the conditions for the complement to be totally

contained within the tiling and shown that the two spirals are self-similar in the monognomonic case. For additional interesting work on gnomons and tiling, see Waldman and Gray [2].

References

- [1] M.J. Gazalé, *Gnomon: From Pharaohs to Fractals*. Princeton University Press (1999).
 [2] Waldman, C.H. and Gray, S.B, Fibonacci, Padovan, & Other Pseudospirals, submitted for publication (2016).

See also: <http://curvebank.calstatela.edu/waldman6/waldman6.htm>.

Appendix: Computer Program

```
function AnamorphicTriangle4NCB
% this program calculates & plots anamorphic whorled triangles &
% spirals
% based the whorled triangle of Gazalé, "Gnomons" on p. 111
% USAGE
% insert a monotonic sequence S=[s1;s2;;...;sn;...] where
% indicated below
% EXAMPLES (OMIT DUPLICATE VALUES)
% S=[1;2;3;5;8;13;21;34;55;...]; % Fibonacci sequence
% S=[1;2;3;5;7;9;12;16;21;28;...]; % Padovan sequence
% S=[1;4;10;14;22;30;35;46;62;72;77;94;105;...]; % Katydid
% sequence
% S=(1:20)'; % linear sequence
% S=primes(50)'; % primes sequence
% S=sort(1+100*rand(20,1)); % Sorted random sequence
%
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S=sort(1+100*rand(20,1));
delS=@(k) (S(k+1)-S(k));

% seed triangle
ztri=[0;(-1+i*sqrt(3))/2;-1;0;0]; % intentionally has 5 points

% trapezoidal gnomons
Z=S(1)*ztri;
kmax=8;
zprev=ztri;
for k=1:kmax
    zgno=[0;-S(k);-
S(k)+delS(k)*exp(i*4*pi/3);delS(k)*exp(i*5*pi/3);0];
    znext=zgno*exp(-i*2*(k-1)*pi/3);
    if k==1
        znext=znext-znext(1)+zprev(1);
    else
        znext=znext-znext(1)+zprev(3);
    end
end
```

```

    end
    Z=[Z znext];
    zprev=znext;
end

% 'super' triangles; previous triangle plus trapezoid
T1=Z(1:4,1);
T2=[Z(4,2);Z(2,1);Z(3,2);Z(4,2)];
T=[T1 T2];
for k=3:kmax+1
    p1=eval(['Z(4,' num2str(k) ')']);
    p2=eval(['Z(4,' num2str(k-1) ')']);
    p3=eval(['Z(3,' num2str(k) ')']);
    this=[p1;p2;p3;p1];
    T=[T this];
end

% for testing, compare
figure;plot(T,'k');axis equal;hold on
figure;plot(Z,'k');axis equal

% PRIMARY SPIRAL
npts=101;
Theta=[7*pi/6,pi/2];
z0=Centroid(Z(:,1));
r=abs(z0-T(1,1));
Pspi=ComplexCircularArc(r,z0,Theta,npts);
for k=2:kmax+1
    Theta=Theta-2*pi/3;
    % get the centroid of the full triangle
    w=T(:,k);
    z0=Centroid(w);
    r=abs(z0-T(1,k));
    zrnd=ComplexCircularArc(r,z0,Theta,npts);
    Pspi=[Pspi;zrnd];
end

% SECONDARY (COMPLEMENTARY) SPIRAL
Sspi=[];
for k=2:kmax+1
    thisrot=angle(Z(1,k)-Z(3,k))+pi/2;
    z0=(Z(1,k)+Z(3,k))/2+exp(i*thisrot)*sqrt(3)/6*abs(Z(1,k)-
Z(3,k));
    r=abs(z0-Z(1,k));
    theta1=angle(Z(1,k)-z0);
    theta2=angle(Z(3,k)-z0);
    if theta1<theta2, theta1=theta1+2*pi; end

```



```

    Theta=[theta1 theta2];
    zrnd=ComplexCircularArc(r,z0,Theta,npts);
    Sspi=[Sspi;zrnd];
end

% TERTIARY (FULL) SPIRAL
npts=101;
Tspi=[];
Theta=[-5*pi/6 pi/2];
z0=Centroid(T(:,1));
r=abs(z0-T(1,1));
f1=ComplexCircularArc(r,z0,Theta,npts);
Tspi=f1;
for k=2:kmax+1
    z0=eval(['T(1,' num2str(k) ')']);
    z0=Centroid(T(:,k));
    theta1=angle(T(2,k)-z0);
    theta2=angle(T(1,k)-z0);
    if theta2<0, theta2=theta2+2*pi; end
    Theta=[theta1 theta2];
    r=abs(z0-T(1,k));
    f1=ComplexCircularArc(r,z0,Theta,npts);
    Tspi=[Tspi;f1];
end

figure;plot(Z,'k');axis equal
hold on
plot(Pspi,'b','LineWidth',2)
plot(Sspi,'Color',[0 .5 0],'LineWidth',2)
v=axis;

figure;plot(Z,'k');axis equal
hold on
plot(Tspi,'r','LineWidth',2)
axis(v)

return

function z=ComplexCircularArc(R,z0,Theta,npts)
% calculate a circular arc of radius R at z=z0 from
Theta=[Theta(1)
% Theta(2)] using npts points in the complex plane

theta=linspace(Theta(1),Theta(2),npts)';
z=R*exp(i*theta)+z0;

return

```

