The Brachistochrone Revisited: A Timely Consideration

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In June 1696 Johann Bernoulli published as a challenge the following problem.

"Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time."

Let v represent the bead's speed and chose point A's coordinates as $(x, y) = (0, L)$ and point B's coordinates as $(a, 0)$. Energy conservation requires that the bead's speed is given by

$$\mathrm{v} = \sqrt{2g(L-y)}\ .$$

The time of travel between Point A and Point B is given by the following integral.

$$T = \int_{x=0}^{x=a} \frac{ds}{\mathrm{v}} = \int_0^a \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{2g(L-y)}}\,dx\ .$$

To simplify the analysis the following "normalized" or scaled variables are introduced.

$$X = \frac{x}{L}\ \ ;\ \ Y = \frac{y}{L}\ \ ;\ \ r = \frac{a}{L}\ \ ;\ \ \tau = \frac{T}{\sqrt{\frac{2L}{g}}}$$

Here $\sqrt{\frac{2L}{g}}$ is the time to drop from rest through a vertical distance L and r is the ratio of the net horizontal displacement to the net vertical displacement.

The solution of the Brachistochrone problem is an inverted cycloid with the bead released from the top left cusp. The constant k is the diameter of the "generating circle" of the cycloid.

$$X = \frac{k}{2}(\theta - \sin\theta)$$

$$Y = 1 - \frac{k}{2}(1-\cos\theta)$$

The bottom of the cycloid must extend below $Y = 0$, so the minimum value of k is 1. The boundary conditions determine the value of k and the maximum value of theta, θ_0.

$$r = \frac{k}{2}(\theta_0 - \sin\theta_0)$$

$$1 = \frac{k}{2}(1-\cos\theta_0)$$

These conditions lead to the following two equations:

$$r = \frac{\theta_0 - \sin\theta_0}{1 - \cos\theta_0} \text{ and } \cos\theta_0 = 1 - \frac{2}{k} \ .$$

The cycloid only has one free parameter. Both k and θ_0 are functions of r. Thus, r is the fundamental independent variable of the Brachistochrone.

The normalized least time is given by

$$\tau_B = \frac{1}{2}\int_0^{\theta_0} \frac{\sqrt{\left(\frac{dX}{d\theta}\right)^2 + \left(\frac{dY}{d\theta}\right)^2}}{\sqrt{1-Y}}\, d\theta = \frac{1}{2}\int_0^{\theta_0} \frac{\frac{k}{2}\sqrt{2-2\cos\theta}}{\sqrt{\frac{k(1-\cos\theta)}{2}}}\, d\theta = \frac{\sqrt{k}}{2}\theta_0$$

$$= \frac{\theta_0}{\sqrt{2(1-\cos\theta_0)}} = \frac{\theta_0\sqrt{r}}{\sqrt{2(\theta_0 - \sin\theta_0)}}$$

Theorem: If $0 < r < \frac{\pi}{2}$, θ_0 is in the interval $(0, \pi)$, if $r > \frac{\pi}{2}$, θ_0 is in the interval $(\pi, 2\pi)$, and if $r = \frac{\pi}{2}$, $\theta_0 = \pi$.

The constant k is given in terms of θ_0 by $k = \dfrac{2}{1-\cos\theta_0}$. From the results stated in the Theorem this can be inverted as follows:

$$\theta_0 = \begin{cases} \cos^{-1}\left(1 - \frac{2}{k}\right) & \text{if } 0 < r < \frac{\pi}{2} \\ \pi & r = \frac{\pi}{2} \\ 2\pi - \cos^{-1}\left(1 - \frac{2}{k}\right) & \text{if } r > \frac{\pi}{2} \end{cases}$$

This result gives two different formulas for r in terms of k. This of course means that r is **not** a function of k .

$$r = \begin{cases} \frac{k}{2}\cos^{-1}\left(1 - \frac{2}{k}\right) - \sqrt{k-1} & \text{if } 0 < r < \frac{\pi}{2} \\ \pi k - \frac{k}{2}\cos^{-1}\left(1 - \frac{2}{k}\right) + \sqrt{k-1} & \text{if } r > \frac{\pi}{2} \end{cases}$$

A parametric plot of this relation is shown in Figure 1.

An interesting feature of the minimizing cycloid occurs when $r > \frac{\pi}{2}$ and θ_0 is in $(\pi, 2\pi)$. The bottom of the cycloid at $\theta = \pi < \theta_0$ is part of the minimizing trajectory. Thus, the curve "over shoots" $y = 0$ and approaches the terminal point of the trajectory from below.

This result is really not surprising. As r increases, the horizontal component of the trajectory dominates the curve. In order to travel this horizontal distance as quickly as possible the vertical drop distance increases to "build up speed". A typical solution with $r > \frac{\pi}{2}$ is shown in Figure 2.

Figure 1

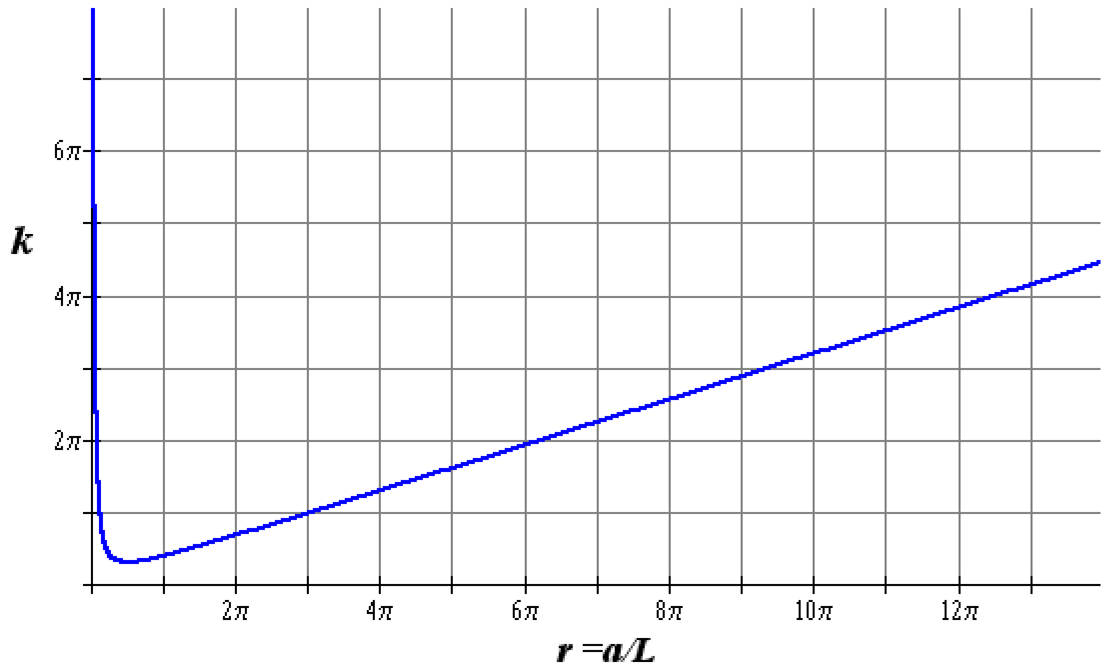
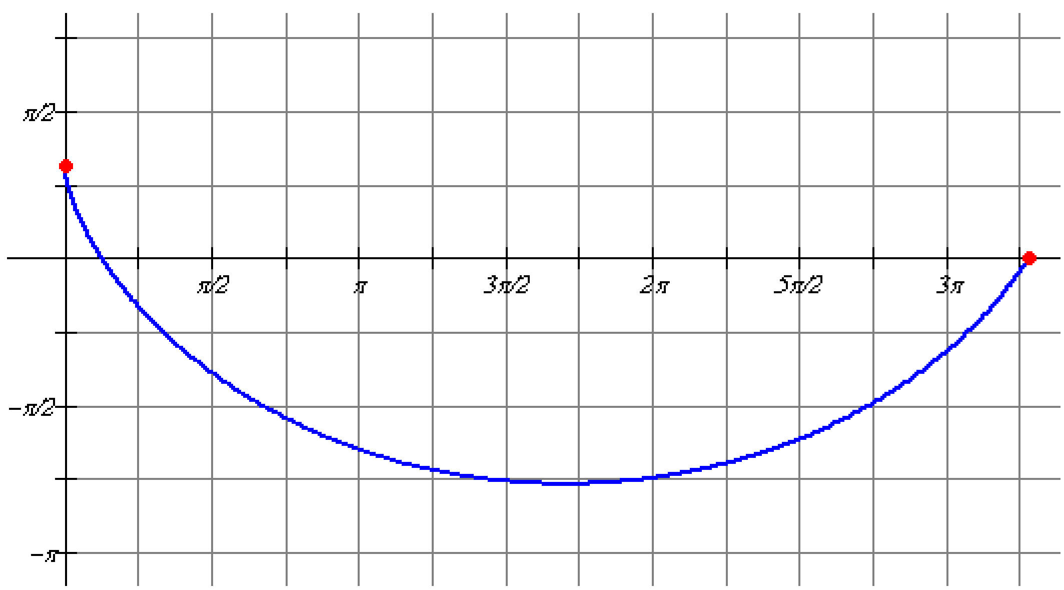


Figure 2



I. Explicit Expansions as a Function of r for the Exact Cycloid Solution:

Asymptotic Expansions for Large r

For $0 \le x < 2$, $\cos^{-1}(1-x) = \sqrt{2x}\left[1+\sum_{j=1}^{\infty}\frac{(1)(3)\cdots(2j-1)}{(2j+1)2^{2j}\,j!}x^j\right]$, and since $k > 1$,

$$\begin{aligned}
r &= \pi k - \sqrt{k}\left[1+\sum_{j=1}^{\infty}\frac{1\cdot 3\cdots(2j-1)}{(2j+1)2^{2j}\,j!}\left(\frac{2}{k}\right)^j\right] + \sqrt{k}\left[1-\sum_{j=1}^{\infty}\frac{1\cdot 3\cdots(2j-3)}{2^j\,j!}\left(\frac{1}{k}\right)^j\right] \\
&= \pi k - \sum_{j=1}^{\infty}\frac{(2j-2)!}{(2j+1)2^{2j-3}\left[(j-1)!\right]^2}k^{-j+\frac{1}{2}} \\
&= \pi k - \frac{2}{3}k^{-\frac{1}{2}} - \frac{1}{5}k^{-\frac{3}{2}} - \frac{3}{28}k^{-\frac{5}{2}} - \frac{5}{72}k^{-\frac{7}{2}} + O\left(k^{-\frac{9}{2}}\right).
\end{aligned}$$

It is reasonable therefore to assume the following expansion for k:

$$k = \frac{r}{\pi} + \sum_{j=0}^{\infty} b_j r^{\frac{1-j}{2}}.$$

The binomial expansion can then be applied to the terms of the form,

$$k^{\frac{1}{2}-\ell} = \left(\frac{r}{\pi}\right)^{\frac{1}{2}-\ell}\left[1+\sum_{j=0}^{\infty}\pi b_j\, r^{-\frac{j+1}{2}}\right]^{\frac{1}{2}-\ell},$$

to give an expansion in powers of $r^{-\frac{1}{2}}$. This yields the following results:

$$k = \frac{r}{\pi} + \frac{2}{3\sqrt{\pi r}} + \frac{\sqrt{\pi}}{5r\sqrt{r}} - \frac{2}{9r^2} + \frac{3\pi\sqrt{\pi}}{28r^2\sqrt{r}} - \frac{4\pi}{15r^3} + \frac{5\sqrt{\pi}(8+3\pi^2)}{216r^3\sqrt{r}} + O\left(r^{-4}\right).$$

.

$$\theta_0 = 2\pi - \frac{2\sqrt{\pi}}{\sqrt{r}} - \frac{\pi\sqrt{\pi}}{3r\sqrt{r}} + \frac{2\pi}{3r^2} - \frac{3\pi^2\sqrt{\pi}}{20r^2\sqrt{r}} + \frac{8\pi^2}{15r^3} - \frac{5\pi\sqrt{\pi}(56+9\pi^2)}{504r^3\sqrt{r}} + \frac{16\pi^3}{35r^4} - \frac{7\pi^2\sqrt{\pi}(352+25\pi^2)}{2880r^4\sqrt{r}} + O\left(r^{-5}\right).$$

$$\tau_B = \frac{T_B}{\sqrt{\frac{2L}{g}}} = \sqrt{\pi r} - 1 + \frac{\pi}{6r} + \frac{\pi^2}{40r^2} - \frac{\pi\sqrt{\pi}}{18r^2\sqrt{r}} + \frac{\pi^3}{112r^3} - \frac{\pi^2\sqrt{\pi}}{30r^3\sqrt{r}} + \frac{\pi^2(128+15\pi^2)}{3456r^4} + O\left(r^{-\frac{9}{2}}\right).$$

Note that in physical units the time of descent scales as $\sqrt{\frac{2L}{g}}\sqrt{\pi\frac{a}{L}} = \sqrt{\frac{2}{g}}\sqrt{\pi a}$. It will be seen in the next section that time of descent increasing like the square root of a is a feature common to a variety of minimum time curves. However, the factor of $\sqrt{\pi}$ associated with the minimizing cycloid is the smallest possible coefficient.

Asymptotic Expansions for Small r

$$k = \frac{4}{9r^2} + \sum_{j=0}^{\infty} c_j r^{2j} .$$

$$k = \frac{4}{9r^2} + \frac{3}{5} + \frac{81}{700} r^2 - \frac{27}{1400} r^4 + \frac{22599}{3080000} r^6 + O(r^8)$$

$$\theta_0 = 3r - \frac{9}{10} r^3 + \frac{729}{1400} r^5 - \frac{729}{2000} r^7 + \frac{2412261}{8624000} r^9 + O(r^{11})$$

$$\tau_B = 1 + \frac{3}{8} r^2 - \frac{81}{640} r^4 + \frac{12393}{179200} r^6 - \frac{1299807}{28672000} r^8 + \frac{2891767311}{88309760000} r^{10} + O(r^{12})$$

As r approaches zero no other path than the minimizing cycloid has an expansion in r for the time of descent smaller than

$$\tau_B = 1 + \frac{3}{8} r^2 + O(r^4) .$$

Expansions about $r = \frac{\pi}{2}$; $s = r - \frac{\pi}{2}$

$$k = 1 + \frac{1}{4} s^2 - \frac{\pi}{16} s^3 + \left(-\frac{1}{24} + \frac{5\pi^2}{256} \right) s^4 + \left(\frac{\pi}{32} - \frac{7\pi^3}{1024} \right) s^5 + \left(\frac{41}{2880} - \frac{7\pi^2}{384} + \frac{21\pi^4}{8192} \right) s^6$$
$$+ \left(-\frac{17\pi}{960} + \frac{5\pi^3}{512} - \frac{33\pi^5}{32768} \right) s^7 + \left(-\frac{247}{40320} + \frac{61\pi^2}{4096} - \frac{165\pi^4}{32768} + \frac{429\pi^6}{1048576} \right) s^8$$
$$+ \left(\frac{257\pi}{24192} - \frac{781\pi^3}{73728} + \frac{1001\pi^5}{393216} - \frac{715\pi^7}{4194304} \right) s^9$$

$$+\left(\frac{21499}{7257600}-\frac{8987\pi^2}{774144}+\frac{9009\pi^4}{1310720}-\frac{1001\pi^6}{786432}+\frac{2431\pi^8}{33554432}\right)s^{10}+O\left(s^{11}\right).$$

$$\theta_0=\pi+s-\frac{\pi}{8}s^2+\left(-\frac{1}{6}+\frac{\pi^2}{32}\right)s^3+\left(\frac{\pi}{12}-\frac{5\pi^3}{512}\right)s^4+\left(\frac{7}{120}-\frac{5\pi^2}{128}+\frac{7\pi^4}{2048}\right)s^5$$
$$+\left(-\frac{311\pi}{5760}+\frac{7\pi^3}{384}-\frac{21\pi^5}{16384}\right)s^6+\left(-\frac{43}{1680}+\frac{3\pi^2}{80}-\frac{35\pi^4}{4096}+\frac{33\pi^6}{65536}\right)s^7$$
$$+\left(\frac{283\pi}{8064}-\frac{191\pi^3}{8192}+\frac{33\pi^5}{8192}-\frac{429\pi^7}{2097152}\right)s^8$$
$$+\left(\frac{4559}{362880}-\frac{3139\pi^2}{96768}+\frac{4037\pi^4}{294912}-\frac{1001\pi^6}{524288}+\frac{715\pi^8}{8388608}\right)s^9$$
$$+\left(-\frac{333199\pi}{14515200}+\frac{48829\pi^3}{1935360}-\frac{61061\pi^5}{7864320}+\frac{715\pi^7}{786432}-\frac{2431\pi^9}{67108864}\right)s^{10}+O\left(s^{11}\right)$$

$$\tau_B=\frac{T_B}{\sqrt{\frac{2L}{g}}}=\frac{\pi}{2}+\frac{1}{2}s-\frac{s^3}{48}+\frac{\pi}{256}s^4+\left(\frac{17}{3840}-\frac{\pi^2}{1024}\right)s^5+\left(-\frac{7\pi}{3072}+\frac{7\pi^3}{24576}\right)s^6$$
$$+\left(-\frac{913}{645120}+\frac{25\pi^2}{24576}-\frac{3\pi^4}{32768}\right)s^7+\left(\frac{1249\pi}{983040}-\frac{29\pi^3}{65536}+\frac{33\pi^5}{1048576}\right)s^8$$
$$+\left(\frac{102113}{185794560}-\frac{983\pi^2}{1179648}+\frac{605\pi^4}{3145728}-\frac{143\pi^6}{12582912}\right)s^9$$
$$+\left(-\frac{17831\pi}{24772608}+\frac{22891\pi^3}{47185920}-\frac{5291\pi^5}{62914560}+\frac{143\pi^7}{33554432}\right)s^{10}+O\left(s^{11}\right).$$

These approximations for k and θ_0 allow for a very accurate graph of the minimizing cycloid for any value of r. In fact, they have been used to generate “computer animations” that display the shape of minimizing cycloid “dynamically” as r changes.

II. Variational Solutions:

A Piece-Wise Linear Trajectory

Consider the path from (0, L) to (a, 0) made up of the following three line segments:

(1) (0, L) to (b, $-D$)
(2) (b, $-D$) to ($b + h$, $-D$)
(3) ($b + h$, $-D$) to (a, 0) .

The variables b and h are constrained to be non-negative numbers which satisfy the inequality that $b+h \le a$. The variable D must be greater than $-L$. The trajectory is illustrated in Figure 3.

Let T_1 be the time on segment 1, T_2 the time on segment 2 and T_3 be the time on segment 3.

$$T_1 = \int_{x=0}^{x=b} \frac{ds}{\mathrm{v}} = \int_0^b \frac{\sqrt{1+\left(y'(x)\right)^2}}{\sqrt{2g(L-y)}}\,dx = \frac{\sqrt{1+\left(-\frac{L+D}{b}\right)^2}}{\sqrt{2g}} \int_0^b \frac{1}{\sqrt{\frac{L+D}{b}x}}\,dx = \sqrt{\frac{2L}{g}}\,\frac{\sqrt{\left(\frac{b}{L}\right)^2+\left(1+\frac{D}{L}\right)^2}}{\sqrt{1+\frac{D}{L}}}$$

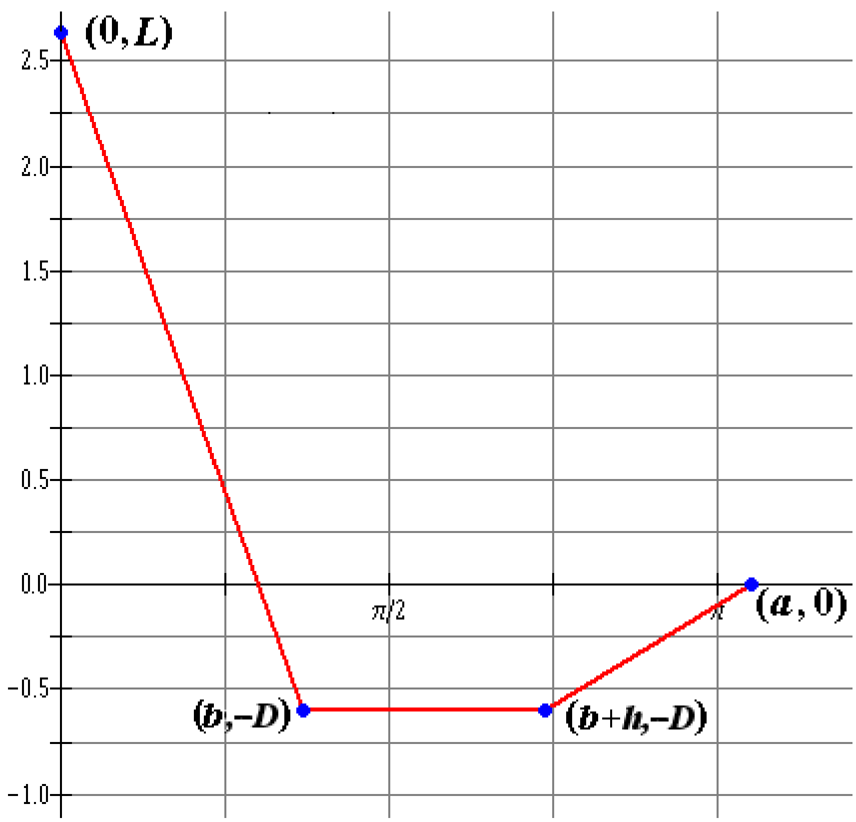
$$T_2 = \frac{h}{\mathrm{v}(\text{at } y=-D)} = \sqrt{\frac{2L}{g}}\,\frac{h}{2L\sqrt{1+\frac{D}{L}}}$$

$$T_3 = \int_{x=b+h}^{x=a} \frac{ds}{\mathrm{v}} = \int_{x=b+h}^{x=a} \frac{\sqrt{1+\left(y'(x)\right)^2}}{\sqrt{2g(L-y)}}\,dx = \frac{\sqrt{1+\left(\frac{D}{a-b-h}\right)^2}}{\sqrt{2g}} \int_{b+h}^{a} \frac{1}{\sqrt{L+D-\frac{D(x-b-h)}{a-b-h}}}\,dx$$

$$= \sqrt{\frac{2L}{g}}\sqrt{1+\left(\frac{a-b-h}{D}\right)^2}\left[\frac{-(a-b-h)}{D}\right]\sqrt{1+\frac{D}{L}-\frac{D(x-b-h)}{L(a-b-h)}}\,\Bigg|_{b+h}^{a}$$

$$= \sqrt{\frac{2L}{g}}\sqrt{1+\left(\frac{a-b-h}{D}\right)^2}\,\mathrm{sgn}(D)\left[\sqrt{1+\frac{D}{L}}-1\right] = \sqrt{\frac{2L}{g}}\sqrt{1+\left(\frac{a-b-h}{D}\right)^2}\left|\sqrt{1+\frac{D}{L}}-1\right|$$

Figure 3



Let T represent the total time along the path, i.e., $T = T_1 + T_2 + T_3$. To simplify the analysis introduce the following "normalized" or scaled variables.

$$\tau = \frac{T}{\sqrt{\dfrac{2L}{g}}} \quad ; \quad r = \frac{a}{L} \quad ; \quad \mathrm{B} = \frac{b}{L} \quad ; \quad \mathrm{E} = \frac{a-b-h}{L} \quad ; \quad \gamma = \sqrt{1 + \frac{D}{L}}$$

As in the analysis of the minimizing cycloid, τ is just the total time measured in units of the time required for a "vertical" fall from rest through a displacement of L. The total vertical displacement along the first segment of the path is given by $L + D = L\gamma^2$. Both B and E are non-negative and must satisfy the constraint $\mathrm{B} + \mathrm{E} \le r$. In terms of these scaled variables the total time along the path is given by

$$\tau(\gamma, \mathrm{B}, \mathrm{E}, r) = \sqrt{\gamma^2 + \left(\frac{\mathrm{B}}{\gamma}\right)^2} + \frac{r - \mathrm{B} - \mathrm{E}}{2\gamma} + \sqrt{(\gamma - 1)^2 + \left(\frac{\mathrm{E}}{\gamma + 1}\right)^2}$$

Straight Line Trajectory (No Free Parameters)

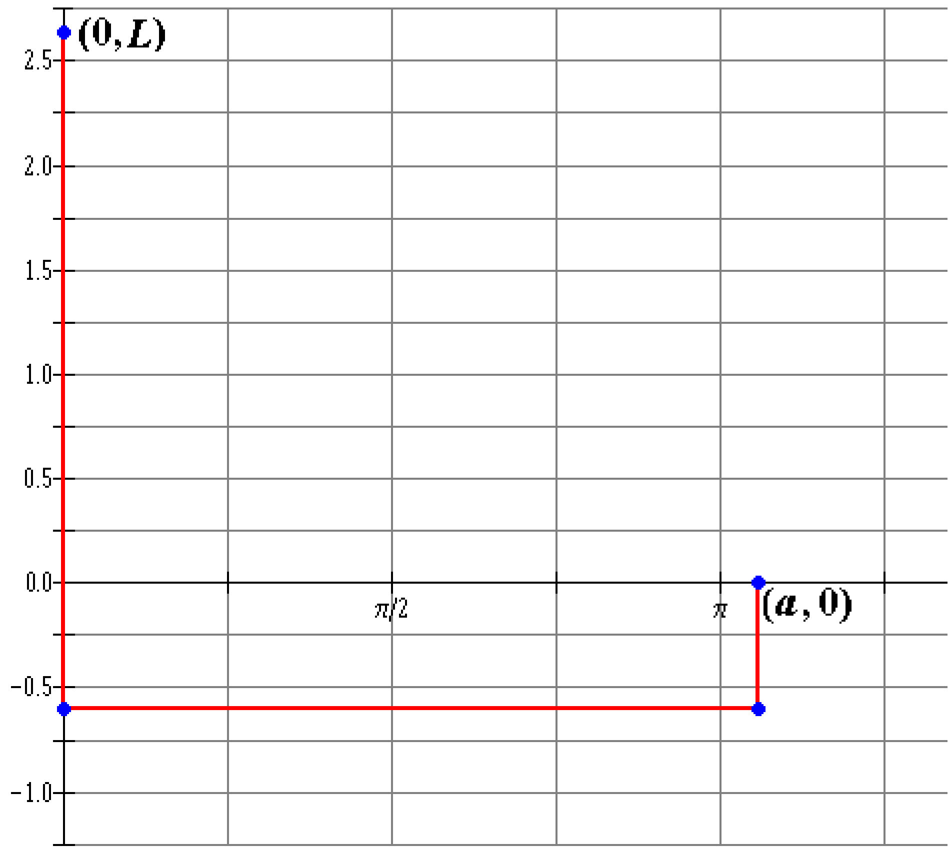
When $\mathrm{B} = r$ and $\gamma = 1\,(D = 0)$, the path is the straight line from (0, L) to (a, 0) .This path has a total time of $\tau(1, r, 0, r) = \sqrt{1 + r^2}$. For small r this can be expanded as $\tau(1, r, 0, r) = 1 + \frac{1}{2}r^2 - \frac{1}{8}r^4 + \frac{1}{16}r^6 + O(r^8)$ which is of course greater than the corresponding time on the cycloid solution to the Brachistochrone which has the expansion

$$\tau_B = 1 + \frac{3}{8}r^2 - \frac{81}{640}r^4 + \frac{12393}{179200}r^6 + O(r^8) .$$

For large r the time on the straight-line path grows nearly linearly in r in contrast to the square root growth of the minimizing path.

An "L" Trajectory (One Free Parameter)

Figure 4



The scaled time along this path is given by

$$\tau(\gamma,0,0,r) = \gamma + \frac{r}{2\gamma} + |\gamma - 1|\,.$$

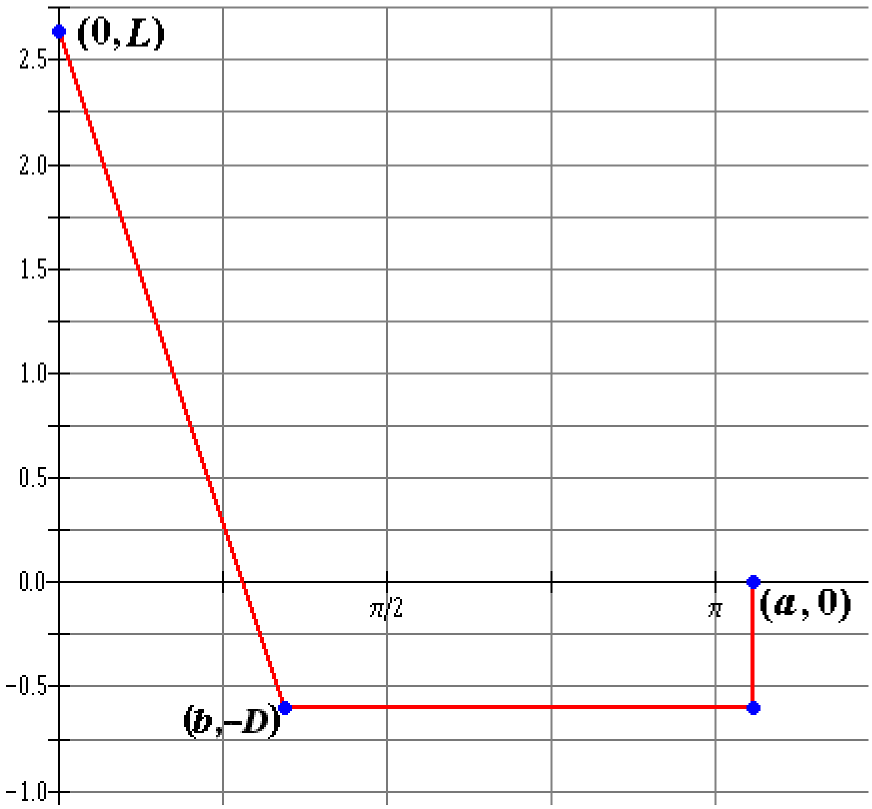
$$\tau_{\min} = \begin{cases} \dfrac{r}{2} + 1 & \text{if } r \le 4 \\ 2\sqrt{r} - 1 & \text{if } r > 4 \end{cases}$$

$$D = L(\gamma^2 - 1) = \begin{cases} 0 & \text{if } r \le 4 \\ L\left(\dfrac{r}{4} - 1\right) = \dfrac{a}{4} - L & \text{if } r > 4 \end{cases}$$

It is interesting to note that $0 < r < \frac{4}{3}$ the minimum total time of the "L" trajectory is actually longer than the time along the straight-line path! For large r it does much better, having a square root dependence on r just like the cycloid solution to the Brachistochrone. Of course 2, the coefficient on the square root of r, is larger than $\sqrt{\pi} = 1.7724538509055\ldots$.

"Left Ramp" Trajectory (Two Free Parameters)

Figure 5



The scaled time along this path is given by

$$\tau(\gamma, \mathrm{B}, 0, r) = \sqrt{\gamma^2 + \left(\frac{\mathrm{B}}{\gamma}\right)^2} + \frac{r - \mathrm{B}}{2\gamma} + |\gamma - 1| \; .$$

One deduces the following relationships:

$$\gamma_0^2 = \begin{cases} 1 & \text{if } r \le 2 + \sqrt{3} \\ \left(2 - \sqrt{3}\right) r & \text{if } r > 2 + \sqrt{3} \end{cases}$$

$$\mathrm{B}_0 = \begin{cases} r & \text{if } r < \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \text{if } \frac{1}{\sqrt{3}} \le r \le 2 + \sqrt{3} \\ r\left(\frac{2\sqrt{3}}{3} - 1\right) & \text{if } r > 2 + \sqrt{3} \end{cases}$$

$$D = L\left(\gamma_0^2 - 1\right) = \begin{cases} 0 & \text{if } r \le 2 + \sqrt{3} \\ \left(2 - \sqrt{3}\right) a - L & \text{if } r > 2 + \sqrt{3} \end{cases}$$

$$\tau_{\min} = \begin{cases} \sqrt{r^2+1} & \text{if } r < \dfrac{1}{\sqrt{3}} \\ \dfrac{\sqrt{3}+r}{2} & \text{if } \dfrac{1}{\sqrt{3}} \le r \le 2+\sqrt{3} \\ \left(\dfrac{\sqrt{6}+\sqrt{2}}{2}\right)\sqrt{r}-1 & \text{if } r > 2+\sqrt{3} \end{cases}$$

For large r the numerical value of the coefficient on the square root of r for the total time is 1.931851652578… . This is better than the "2" for the "L" Trajectory but of course greater than the result for the minimizing cycloid, $\sqrt{\pi} = 1.7724538509055\ldots$

A Parabolic Trajectory (One Free Parameter)

Given the long history of the Brachistochrone and the even longer history of the conic sections the least-time parabolic trajectory is an inherently interesting problem. An arbitrary parabola in the plane has four degrees of freedom: the coordinates of the focus and the placement of the directrix. Thus, determining the least-time parabola between two points means solving a problem with two free parameters. A simpler, but still challenging, problem with a single parameter results if the directrix is constrained to be parallel to the x axis. For convenience, the solution of this problem will still be called the minimizing parabola, while the designation, least-time parabola will be reserved for the solution of the problem with a variable orientation of the directrix. Obviously, the general parabolic path includes the zero-slope directrix as a special case. Hence the time of descent along the least-time parabola will always be shorter than that along the minimizing parabola.

Consider the path of "quickest descent" from $(0, L)$ to $(a, 0)$ when y is a quadratic function of x. Since the given two points must lie on the parabola, there is only one degree of freedom available. This can be taken as the coefficient on x^2, i.e.,

$$y(x) = L + cx^2 - x\left(\frac{L}{a} + ac\right).$$

In terms of the scaled variables $X = x/L$ and $Y = y/L$, with $m = ac$,

$$Y(X) = 1 + \frac{m}{r}X^2 - \frac{X}{r}(1+mr).$$

The parabola has its vertex at $X_{\min} = \dfrac{1+mr}{2m}; Y_{\min} = \dfrac{1}{2} - \dfrac{1}{4mr} - \dfrac{mr}{4} = -\dfrac{1}{4mr}(mr-1)^2.$

The time to drop from from $(0, L)$ to $(a, 0)$ is given by

$$T_p = \int_0^r \frac{\sqrt{1+\left(\dfrac{dY}{dX}\right)^2}}{\sqrt{2gL(1-Y)}} L\,dX = \sqrt{\frac{2L}{g}}\,\frac{1}{2}\int_0^r \frac{\sqrt{1+\left(\dfrac{dY}{dX}\right)^2}}{\sqrt{1-Y}}\,dX\ .$$

$$X = X_{\min}(1-\cos\varphi) = \frac{1+mr}{2m}(1-\cos\varphi); \quad \varphi = \cos^{-1}\left(\frac{1+mr-2mX}{1+mr}\right),$$

$$\tau_p(m,r) = \frac{1}{2}\sqrt{r}\,H(m,r)$$

$$H(m,r) = \frac{1}{\sqrt{m}}\int_0^{\cos^{-1}\left(\frac{1-mr}{1+mr}\right)} \sqrt{1+\left(m+\frac{1}{r}\right)^2\cos^2\phi}\;d\phi$$

For fixed r the minimizing time is found by determining the value of m that minimizes $H(m, r)$.

The value of m, m_0, that minimizes $H(m, r)$ has the following approximate form based on using a fitting polynomial and numerical solutions obtained via Newton's method.

$$m_0(r) = \begin{cases} \dfrac{0.540205760122}{r} + 0.522690675747r - 0.046079190492r^3 & \text{if } r < 1.25 \\ -0.0184200689r^3 + 0.0882426631r^2 - 0.10647589177r + 1.027669882 & \text{if } 1.25 < r < 5.25 \\ 1.5323669440479 - \dfrac{0.660169273467}{\sqrt{r}} - \dfrac{0.223622318846}{r} & r > 5.25 \end{cases}$$

For small r, $\tau_{p,\min} = 1 + 0.414956792r^2 - 0.146487517r^4 + 0.078569364r^6 + O\left(r^8\right)$.

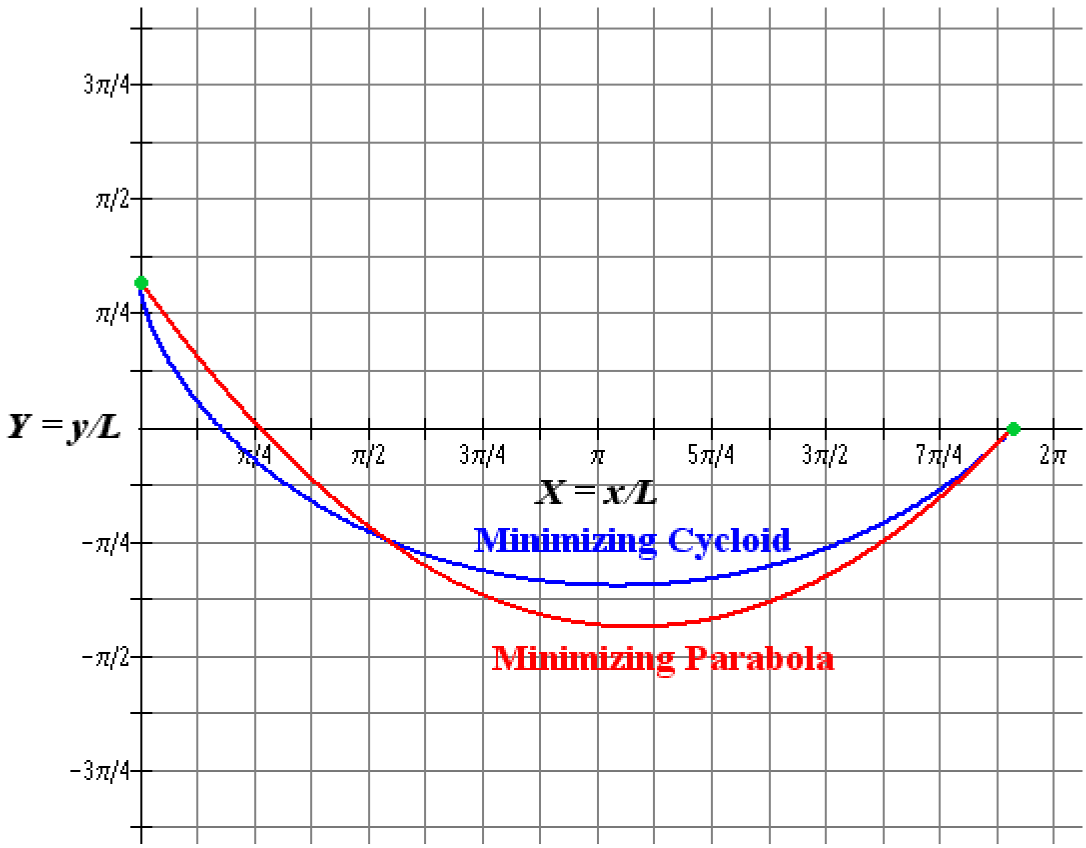
This compares well with the absolute minimum time of the cycloid solution where the coefficient on r^2 is 0.375 and it certainly "beats" the straight line trajectory where the coefficient is 0.5 .

For large r, $\tau_{p,\min} = 1.83421643979696\sqrt{r} - 1.19409691940063 + \dfrac{0.52166717071928}{\sqrt{r}} + O\left(r^{-1}\right)$

The coefficient of the first term of this result when compared to coefficient of the first term of the large r asymptotic time of the minimizing cycloid is too big by 3.48%. However, this is closer than the corresponding result for the "two ramp" trajectory, which is too big by 5.00%.

For large r the minimizing parabola descends about 20% lower than the absolute minimum curve. One could imagine that in the parabola's "race" with the minimizing cycloid it lowers the minimum so as to pick up more speed. But it also picks up more arc length, so in the end it still loses, but not by much. Thus, a parabola can closely approximate the minimum time of descent without being able to closely match the shape of the minimizing cycloid. This appreciable difference in appearance between the two minimizing curves is illustrated in Figure 6 where r = 6.

Figure 6



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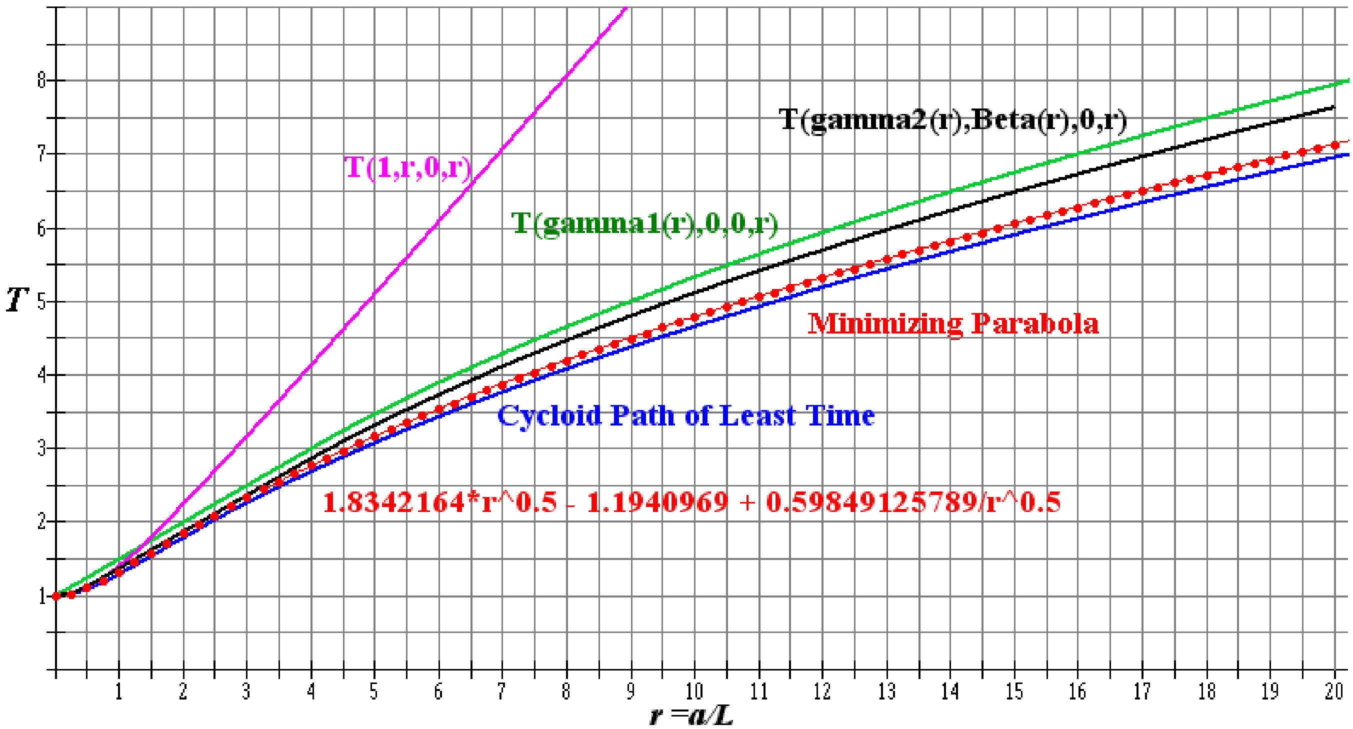
A comparison of the total time of travel between the different variational methods and the minimizing cycloid is displayed in Figure 7. The notation is that T(1, *r*, 0, *r*) is the straight line trajectory, T(gamma1(*r*), 0, 0, *r*) is the "L" trajectory and T(gamma2(*r*), 0, 0, *r*) is the "Left Ramp" trajectory. For the minimizing parabola the discrete points represent the approximate solutions obtained by Newton's method and for $r > 3.5$, the large r asymptotic expression is also plotted.

Figure 7



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