Superconics + Spirals
Cye H. Waldman
Copyright 2017

Introduction

Many spirals are based on the simple circle, although modulated by a variable radius. We have generalized the concept by allowing the circle to be replaced by any closed curve that is topologically equivalent to it. In this note we focus on the superconics for the reasons that they are at once abundant, analytical, and aesthetically pleasing. We also demonstrate how it can be applied to random closed forms.

The spirals of interest are of the general form $z = r(\theta)e^{i\theta}$. A partial list of these spirals is given in the table below:

Spiral	$r(\theta)$	Remarks
Logarithmic	$e^{b\theta}$	b = flair coefficient
Archimedean (generalized)	$a\theta^{1/m}$	$m = 1$, Archimedes $m = 2$, Fermat $m = -1$, Hyperbolic $m = -2$, Lituus
Parabolic	$a\theta^{1/2} + b$	
Cochleoid	$a\sin\theta/\theta$	
Poinsot	$c/\cosh(a\theta)$	

Genesis of the *superspiral* (1)

The idea began when someone on line was seeking to create a 3D spiral with a racetrack planiform. Our first thought was that the circle readily transforms to an ellipse, for example

$$e^{i\theta} = \cos\theta + i\sin\theta \mapsto a\cos\theta + ib\sin\theta$$

Not quite a racetrack, but on the *right* track. The rest cascades in a hurry, as

$$\text{circle} \mapsto \text{ellipse} \mapsto \text{superellipse} \mapsto \text{superconics} \mapsto \text{random}$$

This is subject to the provisions that the random form is closed and has no crossing lines. This also brought to mind Euler's famous derivation,

$$e^{i\pi} + 1 = 0$$

connecting the five most famous symbols in mathematics.

In the present paper we're concerned primarily with the superconics because the mapping is well known and analytic. More information on superconics can be found in Waldman (2016) and submitted for publication by Waldman, Chyau, & Gray (2017). Thus, we take

$$e^{i\theta} \mapsto a\left|\cos^{2/q} u\right| \operatorname{sgn}\left(\cos u\right) + ib\left|\sin^{2p} u\right| \operatorname{sgn}\left(\sin u\right)$$

where $u \in [0, 2\pi]$ is a parametric variable (not to be confused with the polar or tangential angles). At the same time, we note that the parameters a and b are superfluous insofar as they merely stretch the axes and can be affected after the fact.

Figure 1 shows a comparison of two 'normal' spirals (Archimedes above and Fermat below) and a superconics interpretation to the right. Here we have plotted z and $–z$ to create a double spiral in each case.

Figure 2 similarly shows a comparison of two 'normal' spirals (logarithmic above and Cochleoid below) and a superconics interpretation to the right. Here we have plotted z and z^* to create an osculating spiral in each case. You can also use z and $-z^*$ to create alternative osculating spirals.

Figure 3 (upper) shows a random *blob* on the left and an Archimedes spiral interpretation on the right.

Spirals that do not conform to the circular model can also be similarly transformed. Figure 3 (lower) shows a Cornu (or Euler) spiral on the left and superconics interpretation on the right. That interpretation was created as follows,

$$z(t) = C(t) + i S(t)$$
$$= \int_0^t e^{i\frac{\pi}{2}u^2} du = \frac{1+i}{2} \operatorname{erf}\left(\frac{1-i}{2}\sqrt{\pi} \cdot t\right)$$
$$= |z| e^{i\arg(z)}$$

Then we can merely map $e^{i\arg z} \mapsto$ superconics. In this case, however, we can have no expectation that the final result will necessarily look anything like the superconics in question insofar as $\arg(z)$ is not a uniform distribution.

The animations associated with this manuscript show some additional examples of these spirals. The first one (yellow curves) shows an Archimedean spiral with various superconics mappings. The second one (cyan curves) shows a single superconics mapping with a variety of generalized Archimedean spirals (i.e., variable m). Finally, the third one (green curves) shows the Cornu spiral taken through a variety of superconics transforms Keep in mind that the circle is itself a superconics curve, so the 'normal' spiral is a member of the superconics set.

The Matlab computer code in the Appendix will allow you to create your own generalized Archimedean spirals and superconics mappings. Of course, the code may be modified to accommodate any spiral and mapping you would like. The 'elliptic' parameters a and b are also available in the code.
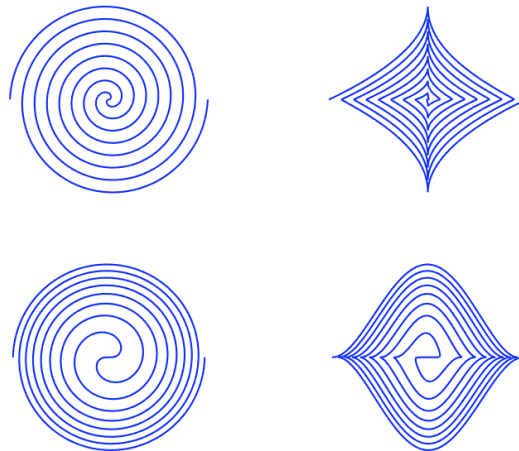

Figure 1. Comparison of 'normal' and transformed spirals: Archimedes (above), Fermat (below).
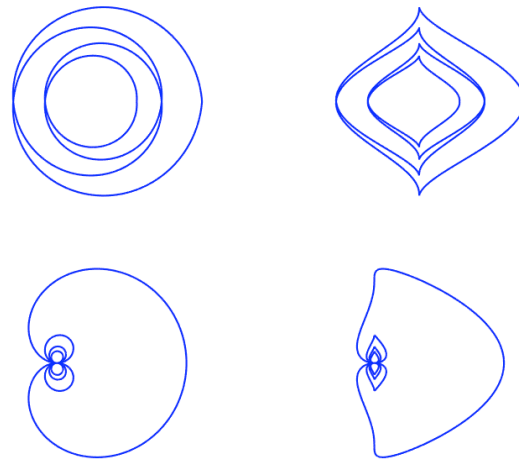

Figure 2. Comparison of 'normal' and transformed spirals: logarithmic (above), cochleoid (below).
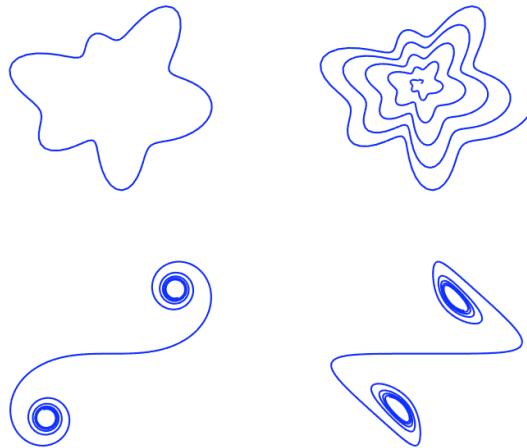
Figure 3. Comparison of 'normal' and transformed spirals: random blob (above), Cornu/Euler (below).

References

Waldman, C.H. (2016). Superconics
http://web.calstatela.edu/curvebank/superconicncb/superconicncb.htm

Waldman, C. H., Chyau, S. Z, and. Gray. S. B. (2017). "Superconics: Hypergeometric Functions," (in preparation).

Appendix: Computer Code

```
function Superspirals4NCB
% create superconics mappings generalized Archimedean spirals
% Copyright July, 2017, Cye H. Waldman
ZsuperE=@(u,p,q,a,b)
a*abs(cos(u).^2).^(1/q).*sign(cos(u))+i*b*abs(sin(u).^2).^p.*sign(sin(u));

u=2*pi*linspace(0,3.5,4001)';
d=1;
m=2*rand; % m=1 --> Archimedes, m=1/2 --> Fermat; try others as well
r=d*u.^(1/m)/2/pi;

% true spiral
p=1/2;q=2;a=1;b=1;
z1=r.*ZsuperE(u,p,q,a,b);

% superconics mapping
p=3*rand;q=3*rand;a=1;b=1;
z2=r.*ZsuperE(u,p,q,a,b);

figure
subplot(1,2,1)
plot([z1 -z1],'b','LineWidth',1)
```

```
title(['m = ' num2str(m)])
axis('equal','tight','off')
subplot(1,2,2)
plot([z2 -z2],'b','LineWidth',1)
title(['[p,q] = [' num2str([p q]) ']'])
axis('equal','tight','off')

return
```