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The Tunnel Effect

The behaviour of particles and particle streams in the presence of very thin potential barriers can be described by wave functions. Especially the case where the particle energy is smaller than the energetic height of the potential barrier is remarkable. In classical physics and normal experience there is no possibility for the particle to overcome the barrier. The fact that it is in contrast possible for the particle to penetrate the barrier in some cases is called “Tunnel Effect”. While at first sight this seems to be a quite abstract scientific subject, the tunnel effect has a wide range of consequences and applications in real life. For example the radioactive decay of atoms is determined by Tunnel Effect and the storage of photos in digital camera storage cards is dependent on the Tunnel Effect.

The mathematical description of particles and particle streams are complex valued wave functions. We look for the wave function psi(x) which is a solution of the one dimensional stationary Schrödinger Equation:

(f1) $$\frac{\partial^2 \psi(x)}{\partial x^2} + \frac{2m}{\hbar^2}[E - E_b(x)]\psi(x) = 0$$

Here Eb(x) stands for a rectangular potential barrier of height Eb extending from x=0 to x=a. m is the particle mass and h bar is the Planck constant divided by 2 pi. To find a solution of the Schrödinger Equation we divide the x-axis into three zones. In zone (1) with x<0 the solution approach for the wave function psi1 is a superposition of an incoming wave and a reflected wave part with factor A. Zone (2) is the region of the potential barrier with 0<x<a. Here we use a general exponential expression psi2 with factors B1 and B2. For x>a in zone (3) there is a transmitted wave psi3 only. It is travelling in the same direction as the incoming wave.

(f2) $$\psi_1(x) := \exp(ikx) + A \cdot \exp(-ikx)$$

(f3) $$\psi_2(x) := B_1 \exp(ipx) + B_2 \exp(-ipx)$$

(f4) $$\psi_3(x) := D \cdot \exp(ikx)$$

By using the continuity of the wave functions and their derivatives at x=0 and x=a we find after some calculation the factors:

(f5) $$A=2k\frac{(k+p)\cdot\exp(-ipa)-(k-p)\cdot\exp(ipa)}{(k+p)^2\exp(-ipa)-(k-p)^2\exp(ipa)}-1$$

(f6) $$B_1=\frac{2k(k+p)\cdot\exp(-ipa)}{(k+p)^2\exp(-ipa)-(k-p)^2\exp(ipa)}$$

(f7) $$B_2=\frac{-2k(k-p)\cdot\exp(ipa)}{(k+p)^2\exp(-ipa)-(k-p)^2\exp(ipa)}$$

(f8) $$D=\frac{4kp\cdot\exp(-ika)}{(k+p)^2\exp(-ipa)-(k-p)^2\exp(ipa)}$$

The parameter k=k1 in functions psi1 and psi3 is related to the free particle energy E by:

(f9) $$k1=\frac{2\pi}{h}\sqrt{2m\,E}$$

m: particle mass, h: Planck constant

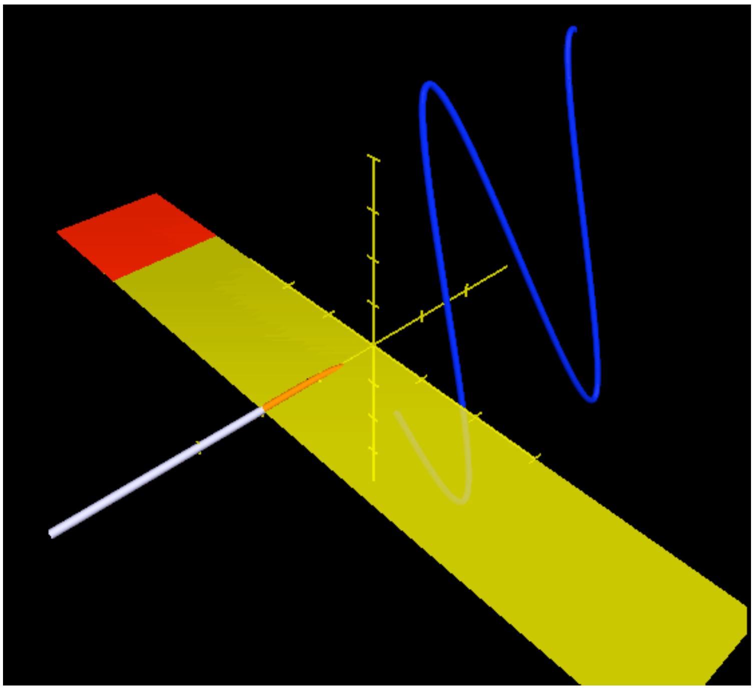
For plotting the wave functions in dependence of E we neglect the constant factors and define:

(f10) $$k=\sqrt{\mathrm{E}}$$

Similarly the wave number p in the barrier zone 2 is set to :

(f11) $$p=\sqrt{(\mathrm{E}-\mathrm{E}_b)}$$

Complex valued wave functions in dependence of coordinate x can be plotted as three dimensional curves. The incoming wave function psi1 is colored blue, psi2 orange and psi3 white. The barrier region zone (2) is depicted as yellow area in the real plane. In the following animation we vary energy parameter E from 10 to 150 for given parameters a = 1 and Eb = 100. A red marker in zone (2) indicates the important condition E < Eb when the classical particle is unable to overcome the barrier.



3D_ANIMATION

For low values of E we observe psi1 (blue) in zone (1) to be pure imaginary. This is due to parameter A of reflected wave which approaches -1 for small E. Transmitted wave psi3 (white) is very near to zero in this case. With increasing E function psi1 (blue) turns its oscillation plane in the direction of the real plane. We see that transmitted amplitude of psi3 (white) in zone (3) is clearly greater than zero even if E<Eb still holds. This is the Tunnel Effect. Increasing E further we observe an intermediate extremum of psi2 (orange) which will be discussed later.

Considering a wave function psi of the form:

(f12) $$\psi(x) = C \cdot \exp(ikx)$$

The particle stream density j of psi is given by [Landau]

(f13) $$j = \frac{\hbar k}{m} C^* C$$

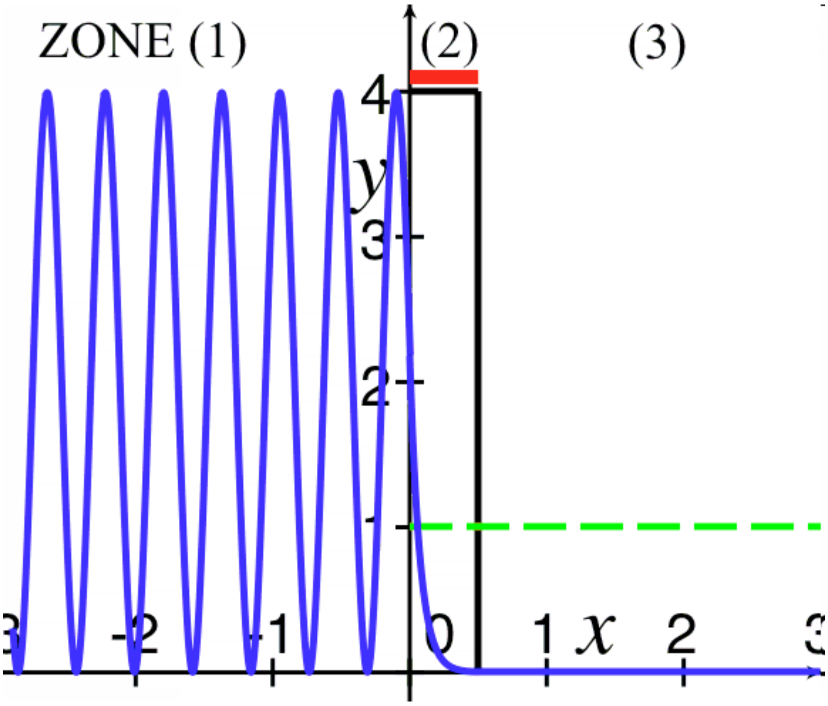
C star is the conjugate complex of C. In other words C star C is the square absolute of C. The transmission coefficient T is given by the ratio of the transmitted and the incoming stream density. Using the incoming part of psi1 and psi3 as transmitted wave we find:

(f14) $$T = \left(\frac{\hbar k}{m} D^* D \right) / \frac{\hbar k}{m} = D^* D = \psi_3{}^* \psi_3$$

This is the probability for a particle to penetrate the potential barrier and being transmitted to zone (3). By plotting psi star psi versus x the transmission coefficient T is directly visible in zone (3). The reflection coefficient R is determined by:

(f15) $$R = 1 - T$$

The distance between T and the green line at y=1 in zone (3) therefore directly shows the reflection coefficient R. In the following animation we vary energy parameter E from E = 50 to E = 280 for given parameters a = 0.5 and Eb = 100.



2D_ANIMATION

The potential barrier between x=0 and x=a is indicated as a box. A red marker line on top of the box in zone (2) appears whenever E is smaller than Eb. The Tunnel Effect with T>0 together with red marker is clearly visible. It is also remarkable that reflection occurs even for E>Eb what is classical not possible. Increasing E further transmission maxima become visible. The maxima correspond to wavelength resonances inside the barrier at:

(f16) $$E - E_b = (n\pi / a)^2 \quad , n = 1,2,3\ldots$$

The first two maxima can be observed at E = 139,5 and E = 257,9 .

Literature

[Landau] L.D. Landau, E.M. Lifschitz, Lehrbuch der theoretischen Physik , Band III Quantenmechanik, 6. Auflage , Akademie Verlag Berlin

[Messiah] Albert Messiah, Quantenmechanik Band 1, de Gruyter, Berlin New York 1976