The Apple of My *i* and Other Sinus Spiral Oddities

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The Apple of My *i* is a member of the sinus spiral family of curves. The sinus spiral is defined by Zwikker (*The Advanced Geometry of Plane Curves and Their Applications*, Dover Press, 1963) as

$$z(u) = \frac{2p}{\sqrt[n]{2}} \frac{1}{(1+iu)^{1/n}} \tag{1}$$

We arrived at this family of curves by a different path. While looking through Weisstein (*CRC Encyclopedia of Mathematics*, CRC Press, Chapman & Hall, 2003) we recognized Cayley's sextic as being related to the Cauchy pulse. That function is defined by Hahn (*Hilbert Transforms in Signal Processing*, Artech House, 1996) as

$$\psi(\tau) = \frac{1}{1-i\tau} \tag{2}$$

where τ is a dimensionless time. This function and all it derivatives are analytic functions (in the Hilbert transform sense). The derivatives are given by

$$\psi(\tau;n) = \psi^{(n)}(\tau) = \frac{d^n\psi}{d\tau^n} = \frac{i^n n!}{(1-i\tau)^{n+1}} \tag{3}$$

Now, we have shown that ψ is amenable to the fractional calculus and that we can write a generalized Cauchy pulse as follows

$$\psi(\tau;n) = \frac{i^n \Gamma(n+1)}{(1-i\tau)^{n+1}} \tag{4}$$

The fractional calculus properties extend to the Fourier transform, of course, and we find that the Fourier transform of the generalized Cauchy pulse is related to the gamma pulse, $\gamma(\tau;n)$, which is essentially the kernel of the gamma function, to wit

$$\begin{gathered} \gamma(\tau;n) = \tau^n e^{-\tau} u(\tau) \\ \int_0^\infty \gamma(\tau;n)\,d\tau = \Gamma(n+1) \end{gathered} \tag{5}$$

where u is the Heaviside function and ϖ (below) is the dimensionless frequency,

$$\mathscr{F}\{\psi(\tau;n)\} = 2\pi\, i^n\, \gamma(\varpi;n)$$
$$\mathscr{F}\{\gamma(\tau;n)\} = i^n\, \psi^*(\varpi;n) \tag{6}$$

But that's another story for another time. Back to the Cauchy pulse: without loss of generality, we can take a normalized pulse that is rotated to the canonical form to be our plane curve generator, namely,

$$\psi(\tau;n) = \frac{1}{(1-i\tau)^{n+1}} \tag{7}$$

This differs from Zwikker's form *only* in scale, rotation, and complex conjugate. The following table, adapted from one in Zwikker (1963), shows the classical plane curves that can be created from the sinus spiral and the corresponding Cauchy coefficients.

Curve	**Zwikker *n***	**Cauchy *n***
Orthogonal hyperbola	-2	-3/2
Straight line	-1	-2
Parabola	-1/2	-3
Tschirnhaus cubic	-1/3	-4
Cayley's sextic	1/3	2
Cardioid	1/2	1
Circle	1	0
Lemniscate	2	-1/2

Table: Properties of sinus spirals.

What we bring to the sinus spiral family is the recognition that with the fractional calculus the parameter n can be literally anything: rational or irrational, real or complex; it makes no difference whatsoever.

Thus we introduce **The Apple of My *i***, a plane curve that is the consequence of the i^{th} derivative of the Cauchy pulse. The curve in Figure 1 is plotted for the range $-\infty < \tau < \infty$. The positive and negative τ arcs start at $(1+0i)$ and move apart from each other only to join up again as they worm their way into the core in a logarithmic spiral as $|\tau| \to \infty$.

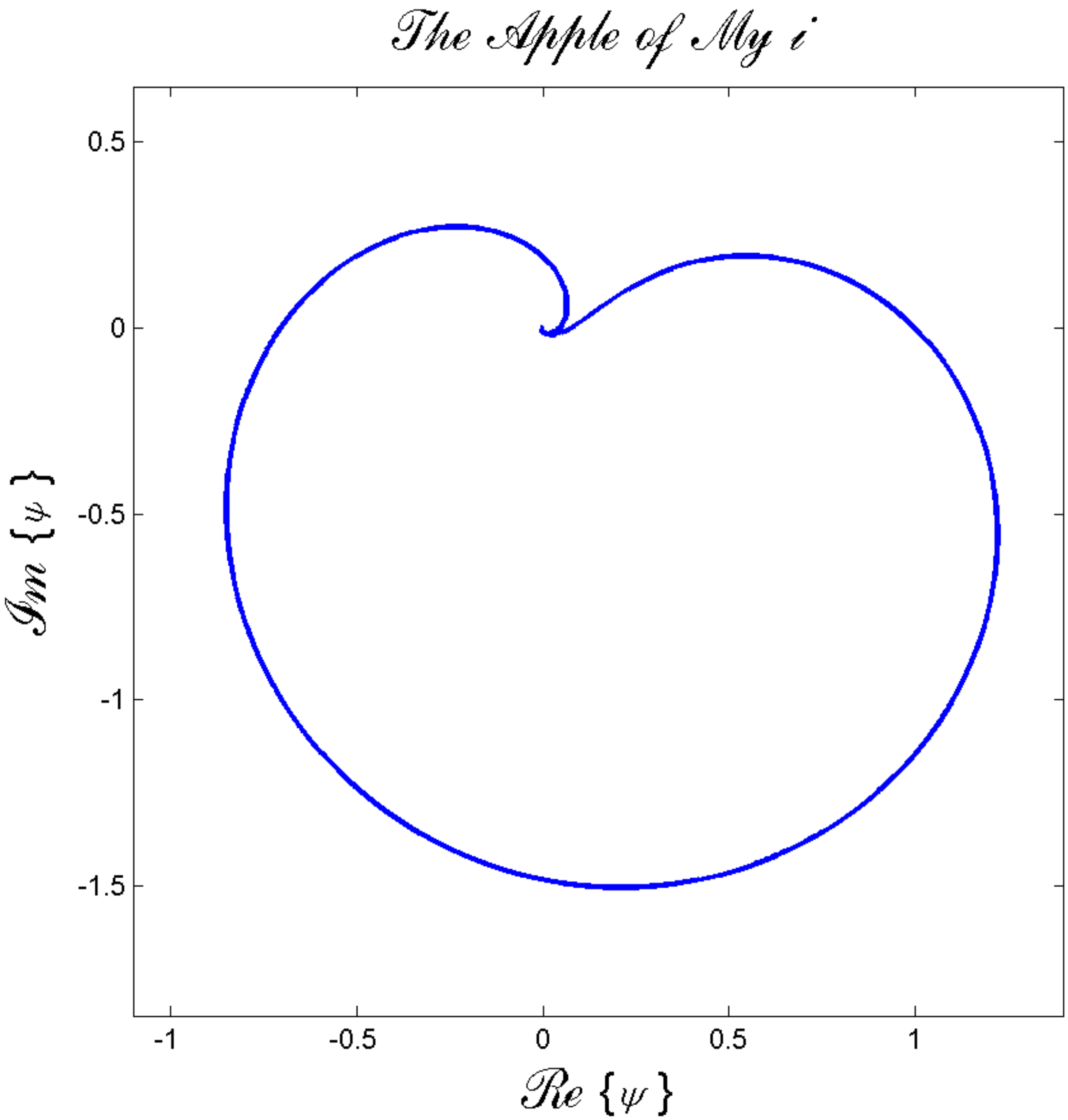

Figure 1: The Apple of My *i*.

The Cauchy pulse can also be expressed in terms of an angular variable by way of the transformation $\tau = \tan\theta$ $\left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$. Thus,

$$\psi\left(\theta; n\right) = \frac{1}{\left(1 - i\tan\theta\right)^{n+1}} = \cos^{n+1}\theta \cdot e^{-i(n+1)\theta} \tag{8}$$

Here we can see that this is a logarithmic spiral (i.e., the exponential term) that is modulated by a sinusoidal term. The attached animation shows the detail of the spiral as we zoom in over 13 orders of magnitude.

Now, in physical problems we are limited to $|\theta| \le \pi/2$, but mathematically we can let θ have extended ranges. In doing so we can grow additional inscribed and/or circumscribed apples at a scale of $\exp(\pm 2\pi)$.

There are several advantages to working with curves in the complex plane. We believe that most of what can be done with the Cauchy pulse would defy description in Cartesian or polar coordinates. In addition, Zwikker (1963) has developed expressions for the arc length, area, and center of gravity (G) in terms of the complex variable $z(u)$. We repeat those equations here for completeness.

$$s = \int |\dot{z}|\, du \tag{9}$$

$$A = \tfrac{1}{2}\int \mathscr{Im}\left\{z^*\dot{z}\right\} du \tag{10}$$

$$z(G) = \frac{\int z\mathscr{Im}\left\{z^*\dot{z}\right\} du}{3A} \tag{11}$$

For **The Apple of My *i*** we find that

$$s = 2\sqrt{2}\sinh(\pi/2) \tag{12}$$

$$A = \frac{\sinh\pi}{4} \tag{13}$$

A closed-form solution for $z(G)$ has eluded us thus far, but numerically it is given by $z(G) = 0.1775 - 0.6022i$.

NOTE ADDED 2 June 2017

A closed form solution for $z(G)$ has been found by Andy Walls and can be found here: https://math.stackexchange.com/questions/2255301/what-is-the-centroid-of-z-frac11-i-taui1-tau-in-infty-infty. The solution is quite a feat and is too complex (pun intended) to go into here. I've simplified his result somewhat by collecting like terms. Without any further ado,

$$z(G) = \frac{\int z\mathscr{Im}\left\{z^*\dot{z}\right\} du}{3A} = \frac{(-1+7i)2^{-i}}{40}\frac{\Gamma^2(i)}{\Gamma(2i)} = 2^{-1-i}(-1+i)\mathrm{B}(i, 3+i) \tag{14}$$

This is one of most interesting derivations I have *ever* seen. Kudos to Andy Walls. I have verified this result numerically.

An excellent case in point for complex variables is the spiral of Cornu. The parametric equations require two integrals, namely, the Fresnel integrals,

$$\begin{aligned} x(t) &= C(t) = \int_0^t \cos\left(\tfrac{\pi}{2}u^2\right)du \\ y(t) &= S(t) = \int_0^t \sin\left(\tfrac{\pi}{2}u^2\right)du \end{aligned} \tag{15}$$

In the complex plane, however, we can achieve a closed-form solution as follows

$$\begin{aligned} z(t) &= C(t) + i\,S(t) \\ &= \int_0^t e^{i\frac{\pi}{2}u^2}du = \frac{1+i}{2}\operatorname{erf}\left(\frac{1-i}{2}\sqrt{\pi}\cdot t\right) \end{aligned} \tag{16}$$

The last equality can be found in Oliver *et al.* (*NIST Handbook of Mathematical Functions*, Cambridge, 2010). Clearly, there are no closed-form solutions in Cartesian space.

NOTE added 26-Jul-12 and corrected on 4/30/13: I have since discovered some additional interesting relationships for the spiral of Cornu from http://www.eecs.berkeley.edu/Pubs/TechRpts/2008/EECS-2008-111.pdf:

$$C(t) + i\,S(t) = t\,{}_1F_1\left(\tfrac{1}{2};\tfrac{3}{2};i\tfrac{\pi}{2}t^2\right)$$

in terms of confluent hypergeometric function. And from Wikipedia:
http://en.wikipedia.org/wiki/Fresnel_integral

$$\varsigma = \sqrt{\frac{\pi}{2}}\,x$$

$$S(x) = \frac{\sqrt{2}}{4}\left(\sqrt{i}\operatorname{erf}\left(\sqrt{i}\,\varsigma\right) + \sqrt{-i}\operatorname{erf}\left(\sqrt{-i}\,\varsigma\right)\right)$$

$$C(x) = \frac{\sqrt{2}}{4}\left(\sqrt{-i}\operatorname{erf}\left(\sqrt{i}\,\varsigma\right) + \sqrt{i}\operatorname{erf}\left(\sqrt{-i}\,\varsigma\right)\right)$$

Note that my solution, Eq. (15), is roughly four times faster since there is but one call to the complex error function.

Some additional examples of curves generated by the Cauchy pulse are shown below. First consider the π^{th} derivative of the Cauchy pulse ($n = \pi$); we get something like a limaçon with an extra internal loop. But if we allow $|\theta| \le 13\pi/2$ we obtain the **Lattice $\boldsymbol{\pi}$** shown in Figure 2. Looking from the circumference in, the image appears to consist of seven overlapping leaves at all scales down to about four orders of magnitude. The attached animation demonstrates this behavior. There are not many seven-leaved plants in nature. However, the list includes cannabis, monk pepper (chaste tree), seven-leaf creeper, and Japanese maple.

Finally, as we allow real number *n*-values to increase, we obtain increasing numbers of nested loops. **The Centurion** shown in Figure 3 for $n = 100$ demonstrates this. There are 50 nested loops down to a scale of 10^{-100}! The attached animation shows the details as we zoom in over a 16 orders of magnitude. For as complicated as this looks, it's much easier to understand when we plot the positive and negative values of θ separately, as shown in Figure 4. These, of course, are just classical sinusoidal spiral curves.

In summary, we have introduced several new dimensions for exploration of plane curves: irrational powers, complex powers, and extended values of θ. These sinus spirals are quite easy to generate. The Matlab code for **The Apple of My *i*** can be expressed in only four lines:

```
psi=inline('1./(1-i*tau).^(n+1)','tau','n');
tau=linspace(-1000,1000,1e6+1)';
n=i;
figure; plot(psi(tau,n)); axis equal
```

Or, in terms of the angular variable:

```
psi=inline('cos(theta).^(n+1).*exp(-i*(n+1)*theta)','theta','n');
theta=linspace(-pi/2,pi/2,1e6+1)';
n=i;
figure; plot(psi(theta,n)); axis equal
```

Obviously, the same code applies for any value of *n* you choose to use.

If you have questions or comments please feel free to contact me at cye@att.net. Go forth and create!

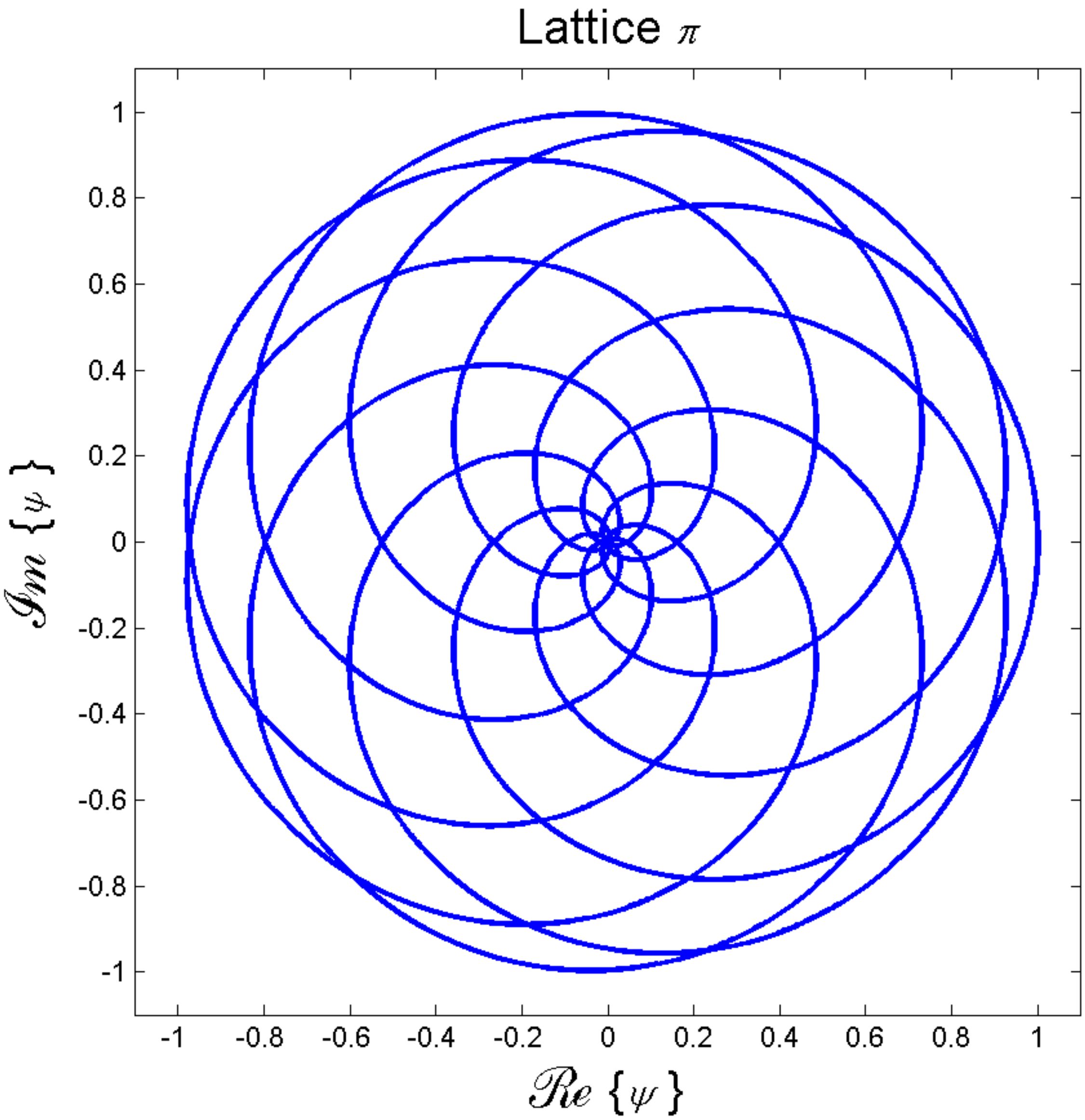

Figure 2: Lattice π.

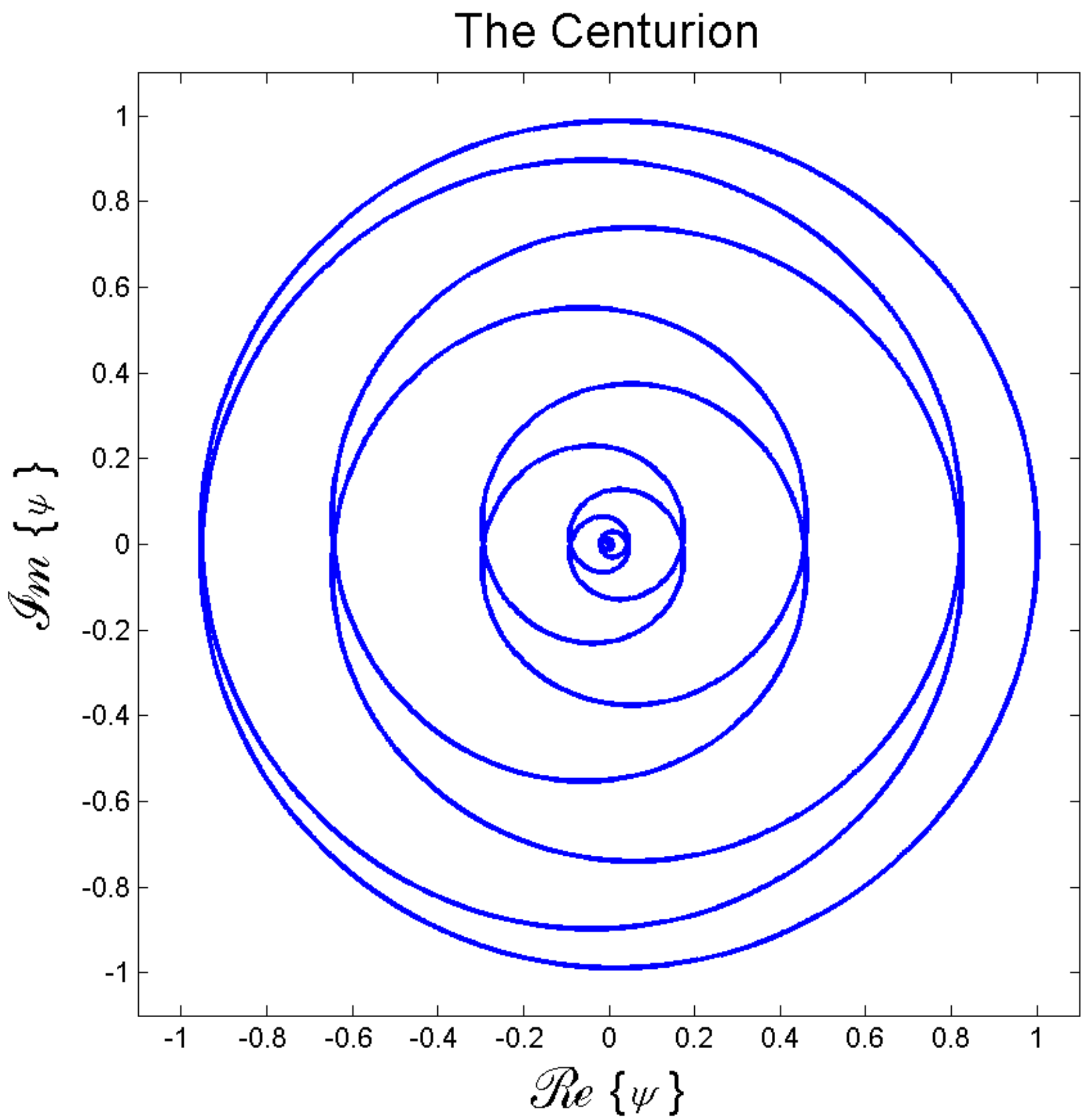

Figure 3: The Centurion.

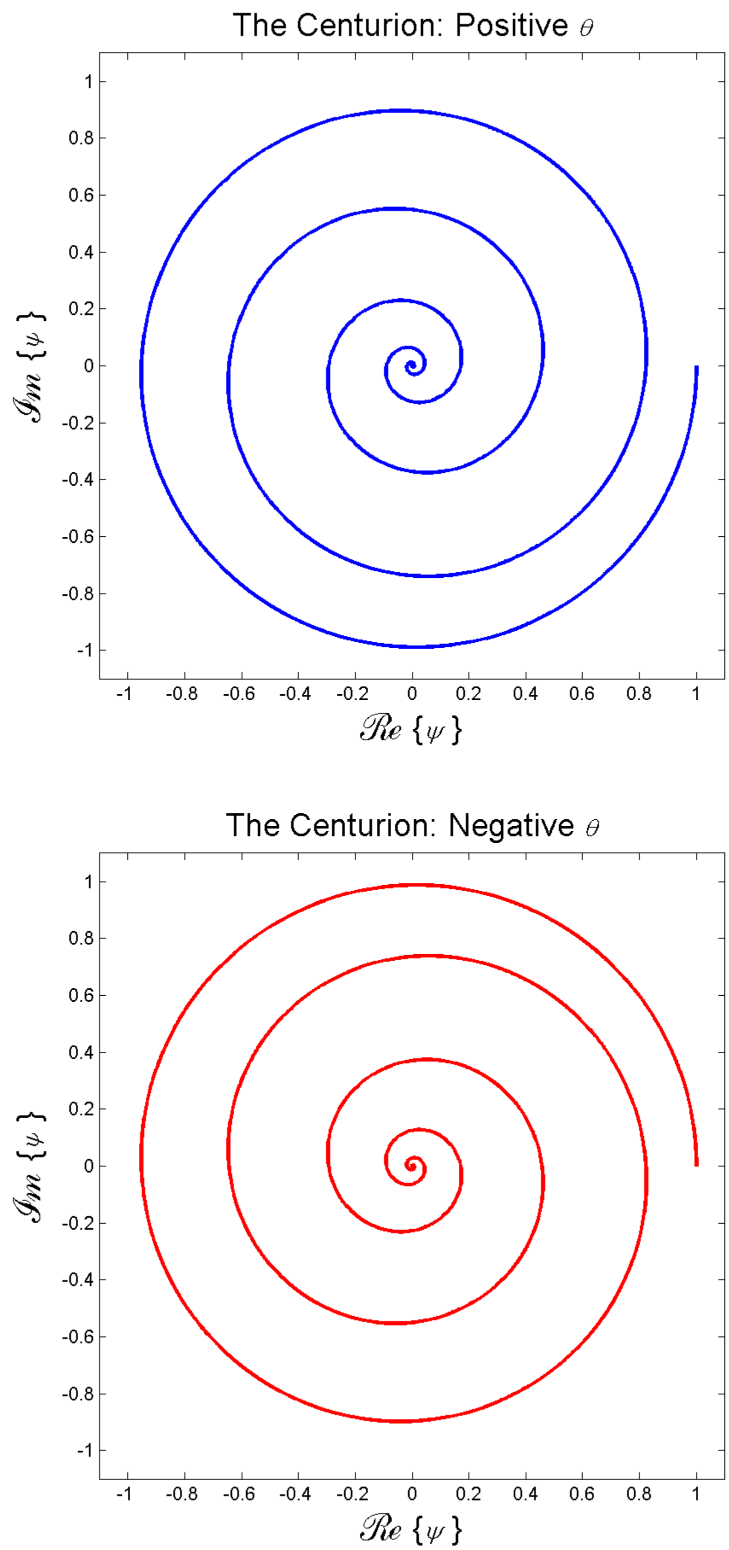

Figure 4: The Centurion: Positive and Negative *θ*.