On Plane Curves in the Complex Plane

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Introduction

The premise of this paper is quite simple. We consider a few techniques for creating plane curves in the complex plane. For the most part, we believe that these methods lead to curves that are at best difficult to express in Cartesian or polar coordinates and more likely impossible to express otherwise in closed form.

Obviously these methods rely heavily on complex numbers. In our traditional science and engineering mathematics coursework, complex variables were invariably a subject apart. Only years later did we begin to think of complex numbers as, well, just numbers. The epiphany came during a reading of Dunham's *Euler: The Master of Us All*. In discussing how complex numbers were legitimized by Euler's discoveries and influence, he says of Euler, "Without apology or embarrassment, he treated these numbers as equal players upon the mathematical stage and showed how to take their roots, logs, sines, and cosines." If this doesn't convince you, I'll leave you with this thought from Johann von Neumann: "In mathematics you don't understand things. You just get used to them." The identity $i^i = e^{-\pi/2}$ itself speaks volumes about how "real" the imaginary number is. There's an interesting story about this equation in Nahin's *An Imaginary Tale: The Story of* $\sqrt{-1}$.

Without any further ado, we introduce three methods for generation of plane curves in the complex plane, by which we mean, plotting the imaginary versus the reals parts of a complex function.

1. Functions of a complex variable
2. Analytic signals created from a function of real variable and its Hilbert transform
3. Fractional derivatives of functions of real or complex variables

In the following sections we shall give some examples of each type. There are several books and Websites that we wish to acknowledge from which we have gained knowledge, inspiration, insight, and sheer enjoyment for this effort. They are listed in the bibliography. Specific references are called out within the text where appropriate.

Complex Functions as Plane Curve Generators

These are simply functions of complex variable; some specific examples follow:

a. The normalized Cauchy pulse, for real τ and real or complex order *n*. This was discussed in detail in "The Apple of My *i*" (http://curvebank.calstatela.edu/waldman/waldman.htm).

$$\psi(\tau;n) = \frac{1}{(1-i\tau)^{n+1}} \tag{1}$$

b. The Cornu spiral, for real τ

$$f(\tau) = \frac{1+i}{2} \operatorname{erf}\left(\frac{1-i}{2}\sqrt{\pi}\,\tau\right) \tag{2}$$

c. The Bessel function of complex argument z and/or order ν

$$f(z;\nu) = J_\nu(z) \tag{3}$$

Figure 1 shows an example of a Bessel function curve for $\nu = 1/2 - 50i$ and $z = (1-i)s$ for $s = [0,24]$. The figure is stretched out normal to complex plane in order to demonstrate how the spiral radius first increases and then decreases. The animation shows how the spirals change over a range of Bessel function order. The radial magnitude is highly variable from frame to frame and has been normalized for the animation.

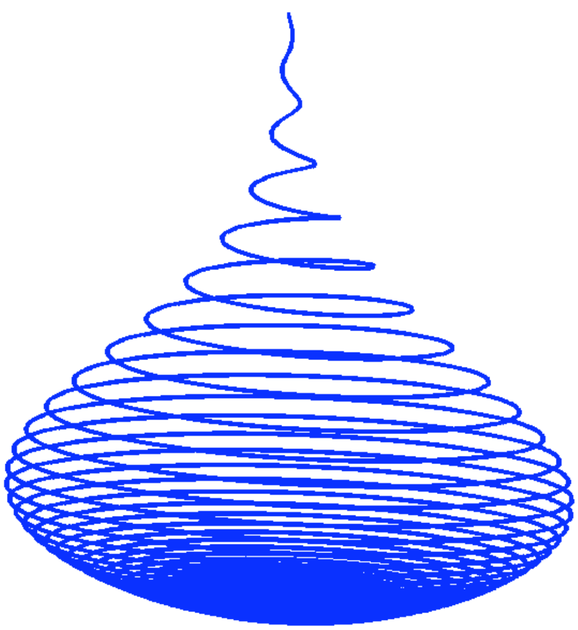

Figure 1: Example curve for complex Bessel function.

Analytic Signals as Plane Curve Generators

The analytic signal consists of a real part, say $f(x)$, and an imaginary part that is its Hilbert transform. That is, the imaginary part is a version of the original real sequence with a 90° phase shift. Thus,

$$f(x) \Rightarrow f(x) + i\,\mathcal{H}\{f(x)\} \tag{4}$$

In this way we can start with virtually any real function and end up with a complex function suitable for plane curve generation.

We present an example for the weighted normalized Hermite polynomial. Specifically, one step at a time, the Hermite polynomial is defined as

$$\begin{gathered}
\mathrm{H}_n(\tau) = (-1)^n e^{\tau^2} \frac{d^n}{d\tau^n} e^{-\tau^2} \qquad -\infty < \tau < \infty \\
\int_{-\infty}^{\infty} \exp\left(-\tau^2\right) \mathrm{H}_n(\tau) \mathrm{H}_m(\tau) d\tau = \begin{cases} 0 & m \neq n \\ \sqrt{\pi}\, 2^n n! & m = n \end{cases}
\end{gathered} \tag{5}$$

The weighted Hermite polynomial is defined in Hahn's *Hilbert Transforms in Signal Processing*. We use the following form to normalize the integral in Eq. (5),

$$u_n(\tau) = \frac{1}{\sqrt{\sqrt{\pi}\, 2^n n!}} e^{-\tau^2} \mathrm{H}_n(\tau) = \frac{(-1)^n}{\sqrt{\sqrt{\pi}\, 2^n n!}} \frac{d^n}{d\tau^n} e^{-\tau^2} \qquad -\infty < \tau < \infty \tag{6}$$

Figure 2 shows the plane curve for the analytical signal of the Hermite polynomial for n = 50. This looks remarkably like the plane curve obtained from the Cauchy pulse with n = 100. We set out to see if they are indeed the same. We have previously shown the following asymptotic solution for the Cauchy pulse for large n (see Waldman, "The Compleat Gamma Pulse" (Web address forthcoming)

$$\lim_{m \to \infty} \psi(\tau; m) = e^{-im\tau} e^{-\frac{\tau^2}{2m}} = e^{-i\sqrt{m}x} e^{-\frac{x^2}{2}} \tag{7}$$

where $x = \tau / \sqrt{m}$. Taking the asymptotic solution for the Hermite polynomials from Spanier and Oldham's *An Atlas of Functions* and expressing them in the normalized form, as per Eq. (6),

$$\lim_{n \to \infty} u_n(\tau) = \begin{cases} \dfrac{(-2)^{n/2} (n-1)!!}{\sqrt{\sqrt{\pi}\, 2^n n!}} \mathrm{e}^{-\tau^2/2} \cos\left(\tau\sqrt{2n+1}\right) & \text{even } n \\[2ex] \dfrac{(-2)^{(n-1)/2} \sqrt{\dfrac{n}{2}}\, n!!}{\sqrt{\sqrt{\pi}\, 2^n n!}} \mathrm{e}^{-\tau^2/2} \sin\left(\tau\sqrt{2n+1}\right) & \text{odd } n \end{cases} \tag{8}$$

From here it is straightforward to that show that for even n and $m = 2n$ that Eqs. (7) and (8) agree to within a constant value of

$$C = \frac{(-2)^{n/2} (\mathrm{n}-1)!!}{\sqrt{\sqrt{\pi}\, 2^n n!}} \quad \left(= -0.2157 \text{ for } n = 50\right) \tag{9}$$

This is in perfect agreement with the numerical results. Other values of *n* will lead to agreement to within rotations of $\pm\pi/2$ in the complex plane for odd *n*. The animation shows the evolution of the Hermitian curves for *n* = 1:50.

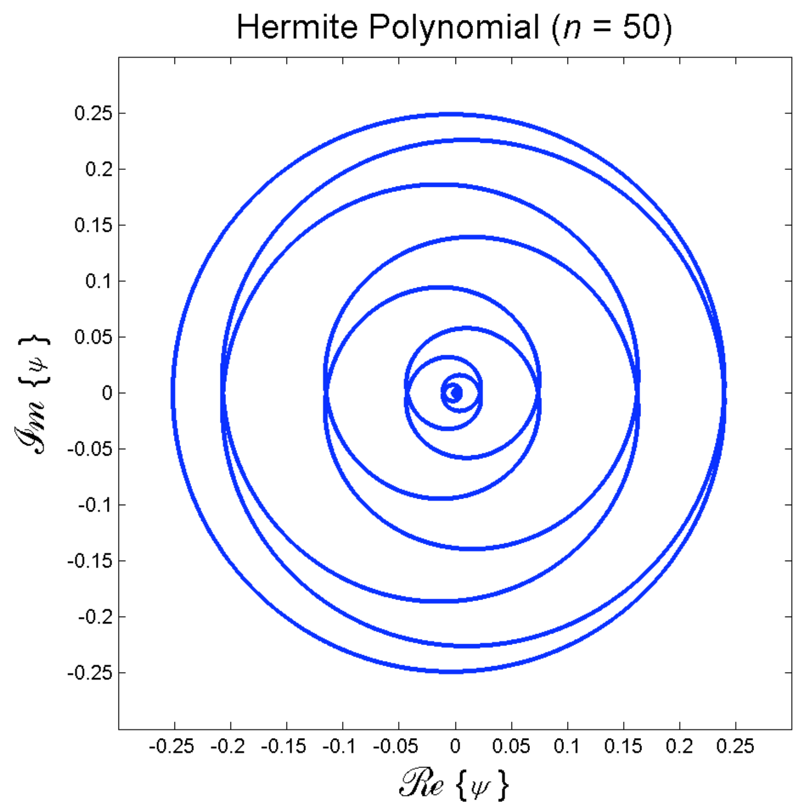

Figure 2: The analytic signal of the weighted normalized Hermite polynomial (*n* – 50).

Figure 3 shows the three-dimensional version of Figure 2. The plane curve of Figure 2 is shown in blue on the x-y plane and the real and imaginary parts of the function are shown on the vertical planes, in red and green, respectively. The vertical axis here is $z = \tan^{-1}\tau$.

Fractional Differintegration as Plane Curve Generators

A discussion of fractional calculus is beyond the scope of this paper. The Wikipedia article on Fractional Calculus (http://en.wikipedia.org/wiki/Fractional_calculus) gives a brief, but accessible, overview of the subject. The book by Oldham and Spanier, *The Fractional Calculus*, gives a detailed exposition with many examples of differintegrable functions. The paper by Lavoie *et al*. (1976) goes into detail on the relationship between fractional derivatives and special functions of mathematical physics.

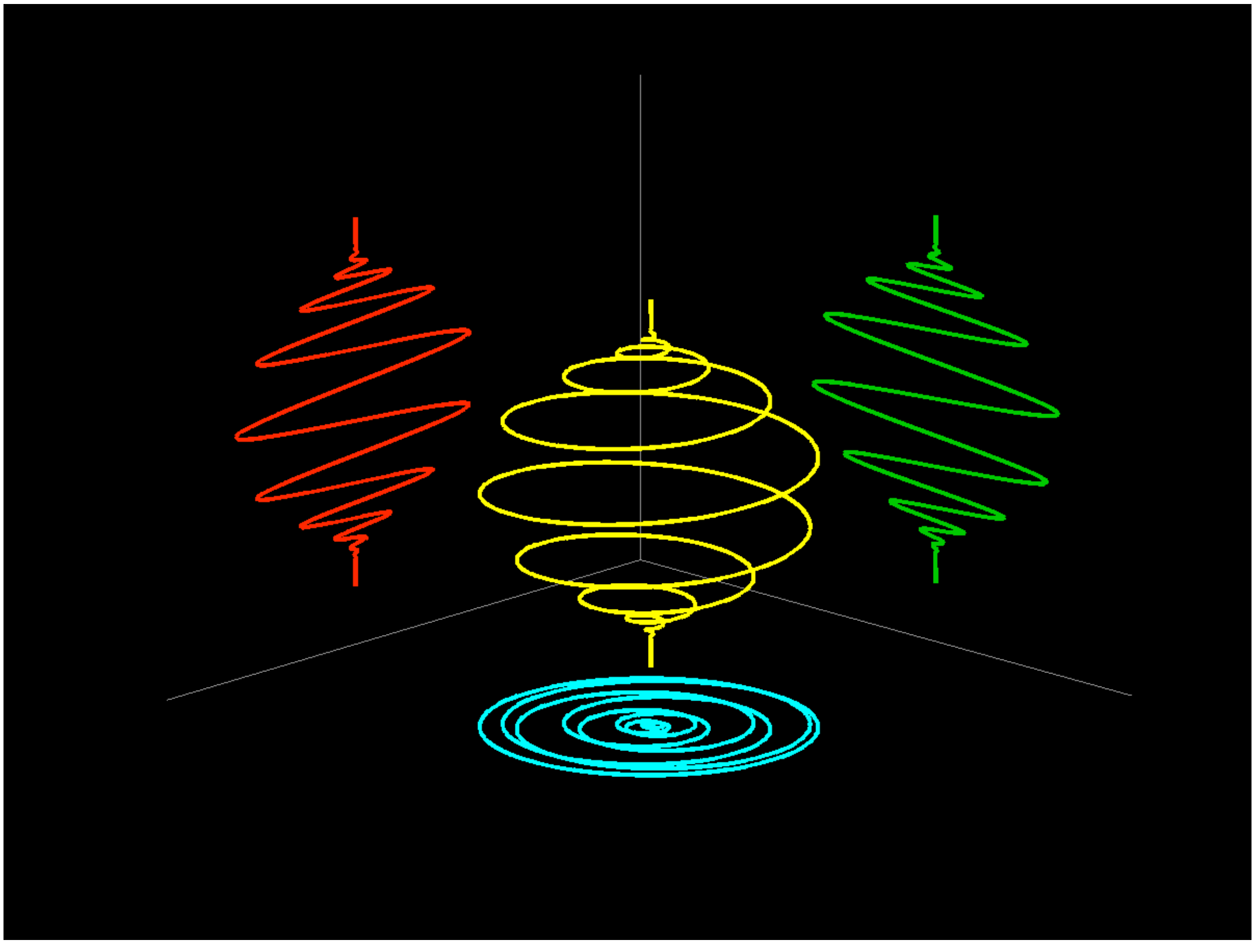

Figure 3: The three-dimensional version of Figure 2.

There are many paths to complex functions with fractional calculus. Specifically, we can find real or complex differintegrals of real or complex functions. The general form of the differintegral is

$$\frac{d^q}{dz^q} f(z) \tag{10}$$

where q and z can be real, imaginary, or complex. Negative q indicates integration. Many authors use a simplified notation such as $D^q f(z)$. Of course, the possibilities here are infinite; we'll just look at a few examples using examples from Oldham and Spanier (1974). Incidentally, they have collected a rather large number of semi-differintegrals $(q = \pm 1/2)$, but we will be using more general forms.

The Arrhenius Form

$$\frac{d^q}{dz^q}\left\{\frac{\exp\left(-1/x\right)}{x^{1-q}}\right\}=\frac{\exp\left(-1/x\right)}{x^{1+q}} \tag{11}$$

We have applied this moniker because the $\exp\left(-1/x\right)$-term is reminiscent of the temperature dependence of chemical reactions. After playing with this function for a while we recognized it as the gamma pulse, which we have studied in detail previously (see, for example, Web site TBD). The gamma pulse is defined as

$$\gamma\left(\tau;n\right)=\tau^n e^{-\tau}u\left(\tau\right) \tag{12}$$

Applying the transformation $\tau=1/x$, we find that

$$\left[-\tau^2\frac{d}{d\tau}\right]^q\gamma\left(\tau;1-q\right)=\gamma\left(\tau;1+q\right) \tag{13}$$

Many examples of the gamma function are given on the referenced Web page.

The Trigonometric and Hyperbolic Sine Forms

$$\begin{aligned}\frac{d^q}{dz^q}\sin\sqrt{x}&=\frac{\sqrt{\pi}}{2}\left(\frac{2}{\sqrt{x}}\right)^{\frac{1}{2}-q}J_{\frac{1}{2}-q}\left(\sqrt{x}\right)\\ \frac{d^q}{dz^q}\sinh\sqrt{x}&=\frac{\sqrt{\pi}}{2}\left(\frac{2}{\sqrt{x}}\right)^{\frac{1}{2}-q}I_{\frac{1}{2}-q}\left(\sqrt{x}\right)\end{aligned} \tag{14}$$

where J and I are the ordinary and hyperbolic (modified) Bessel functions. Figure 4 shows examples of spirals from both forms.

The Bessel Forms

$$\left[\frac{d}{zdz}\right]^q z^\nu J_\nu\left(z\right)=z^{\nu-q}J_{\nu-q}\left(z\right) \tag{15}$$

Figure 5 shows the double spiral that accrues with negative and positive (real) values of z. the associated animation is shown in three-dimensions to underscore the differences in magnitude of and direction of rotation. Note that the negative z-values range from 10^{-7} to zero while the positive values range from zero to 17, yet their spirals are of equal magnitude. The Bessel function grows very rapidly in the negative half-plane.

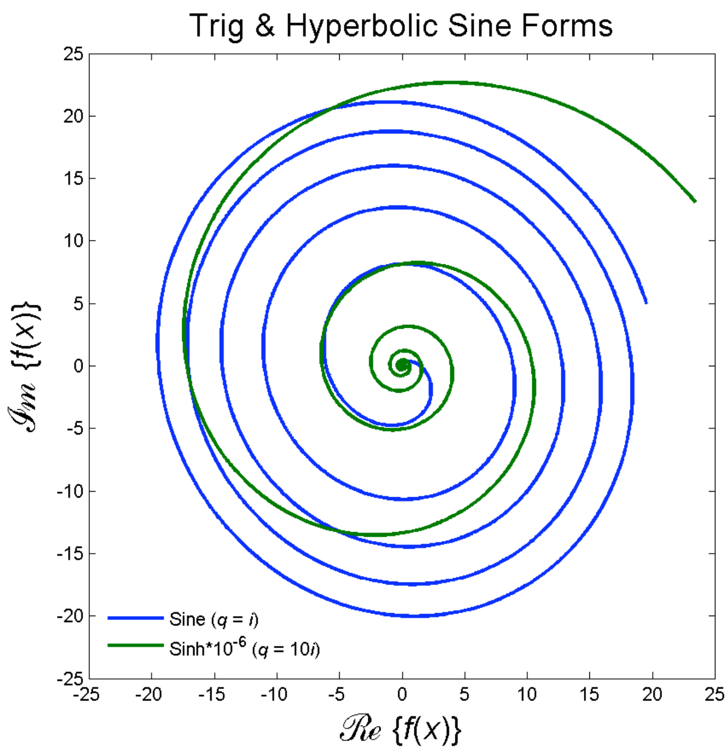

Figure 4: Example spirals from the sine froms ($q = i$).

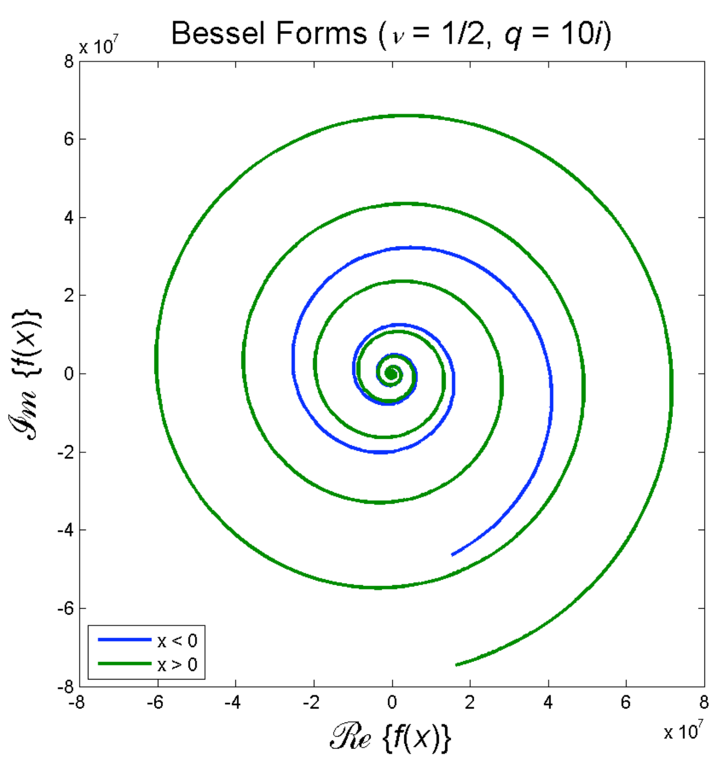

Figure 5: Example spirals from the Bessel froms ($\nu = ½$, $q = 10\ i$).

Concluding Remarks

We have shown that there are many paths to creating plane curves in the complex plane and have barely scratched the surface. There are plenty of opportunities here yet to be explored. In addition to the methods shown here, you might consider the Fourier transform and its various cousins, but we have not had much success with that beyond the beautiful Fourier transform pair of the Cauchy and gamma pulses. The other observation I have made is that we seem to be locked into spirals perhaps someone smart than I can either break out of this of this endless spiral or prove why it is so.

On that note we leave it your own creativity. If you have questions or comments please feel free to contact me at cye@att.net. Go forth and create!

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