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The Incomplete Gamma Function for L-A Curves with Integer Parameter

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In this technical note we develop closed-form solutions for the class of log-aesthetic curves that are defined by the incomplete gamma function with integer parameter. This note follows the analysis and conventions of Ziatdinov *et al*. (2012).

Here is a quick review of how we arrived at the solution in terms of the incomplete gamma function. From the definition of log-aesthetic curve we have for case of $\alpha \notin [0,1]$

$$\begin{aligned} z(\psi) &= \int_0^{\psi} \rho(\theta) e^{i\theta}\, d\theta \\ \rho(\theta) &= \left[(\alpha-1)\lambda\theta+1\right]^{\frac{1}{\alpha-1}} \end{aligned} \tag{1}$$

Make the following transformation of variables:

$$\begin{aligned} t &= -i\frac{1}{(\alpha-1)\lambda}\left[(\alpha-1)\lambda\theta+1\right] \\ \theta &= i\,t-\frac{1}{(\alpha-1)\lambda}; \qquad d\theta = i\,dt \\ \rho(t) &= \left[i(\alpha-1)\lambda t\right]^{\frac{1}{\alpha-1}} \\ a &= \frac{\alpha}{\alpha-1} \end{aligned} \tag{2}$$

Substitute into Eq. (1) to get the result in terms of the incomplete gamma function

$$\begin{aligned} z(\psi) = x(\psi)+i\,y(\psi) &= \int_{t(0)}^{t(\psi)} \left[i(\alpha-1)\lambda t\right]^{a-1} e^{i\left(i\,t-\frac{1}{(\alpha-1)\lambda}\right)} i\,dt \\ &= i\left[i(\alpha-1)\lambda\right]^{a-1} e^{-\frac{i}{(\alpha-1)\lambda}} \int_{t(0)}^{t(\psi)} t^{a-1}e^{-t}dt \\ &= i\left[i(\alpha-1)\lambda\right]^{a-1} e^{-\frac{i}{(\alpha-1)\lambda}} \Gamma(a,t)\Big|_{t(\psi)}^{t(0)} \end{aligned} \tag{3}$$

We now confine our attention to the case of integer values of *a*. These are consistent with the special values of α [Ziatdinov *et al.* (2012)] that also lead to solutions in terms of polynomials. We note that for integer values of *a* the incomplete gamma function reduces to a polynomial in *t* [see, for example, Olver *et al.* (2010)], to wit

$$\Gamma(a,t) = (a-1)!e^{-t}\sum_{k=0}^{a-1}\frac{t^k}{k!} \tag{4}$$

The closed-form solution for $z(\psi)$ can now be written as

$$z(\psi) = i^a\left[(\alpha-1)\lambda\right]^{a-1}(a-1)!\, e^{-\frac{i}{(\alpha-1)\lambda}}\, e^{-t}\sum_{k=0}^{a-1}\frac{t^k}{k!}\Bigg|_{t(\psi)}^{t(0)} \quad a \geq 2 \tag{5}$$

Also, note that the product of the exponentials may be written alternatively as $e^{-i/(\alpha-1)\lambda}e^{-t} = e^{i\theta}$.

The table below shows the polynomial terms for the first few values of *a*.

a	$\sum_{k=0}^{a-1}\frac{t^k}{k!}$
2	$1+t$
3	$1+t+\frac{t^2}{2!}$
4	$1+t+\frac{t^2}{2!}+\frac{t^3}{3!}$
5	$1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}$
$\vdots$	$\vdots$
10	$1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}+\frac{t^6}{6!}+\frac{t^7}{7!}+\frac{t^8}{8!}+\frac{t^9}{9!}$
	Etc.

The proposed solution has two advantages over those in Ziatdinov *et al*. Specifically, the entire solution is contained in a single complex variable with a single summation. In addition, due to the transform of variables, the polynomial is greatly simplified and consists only of powers of *t* with constant coefficients, i.e., independent of *a* and λ.

Figures 1 - 6 show results for *a*-values in ascending powers of 2, i.e., $a \in [2,4,8,16,32,64]$, for various values of λ, $\lambda \in [0.01, 0.05, 0.1, 1.0]$. The computation was performed with the recommended lower limit for the integration given by Ziatdinov *et al*. The figures compare solutions by three different methods: a. direct numerical simulation of Eq. (1) using the trapezoidal method; b. solution in terms of the incomplete gamma function, Eq. (3) (bottom), using the asymptotic form of the incomplete gamma function series expansion; and c. solution in terms of the polynomial, Eq. (5). In all cases, the results are indistinguishable at the scale of the figures. For values of $a \gtrsim 75$ the incomplete gamma function and polynomial methods run into overflow problems, but the direct numerical simulation has no such problems (the largest value we tried was $a = 2^{12}$).

The aggregate time for the computation all 24 cases for each method, representing a broad range of a and λ values, are shown in the table below. Calculations were performed with 10^5 and 10^6 θ–values. The calculations were carried out on Pentium 6-core i7 3.20 GHz computer using Matlab. The exceptional speed of the DNS can be attributed to the Matlab function `cumtrapz`; this is in lieu of 10^5-10^6 *for* loop calculations.

Time (s) to calculate all 24 cases $a \in [2,4,8,16,32,64]$; $\lambda \in [0.01, 0.05, 0.1, 1.0]$		
Method	$\Delta\theta = \mathcal{O}(10^{-5})$	$\Delta\theta = \mathcal{O}(10^{-6})$
DNS: $\int_0^\psi \rho(\theta) e^{i\theta}\, d\theta$	0.23	3.25
IGF: $f(a,\lambda)\cdot\Gamma(a,t)$	12.3	128.6
Poly: $f(a,\lambda)\cdot e^{-t}\sum_{k=0}^{a-1}\frac{t^k}{k!}$	1.65	14.2

The figures are at the end of the document.

References

Olver, F.W.J., Lozier, D.W., Boisvert, R.F., and Clark, C.W., (2010). *NIST Handbook of Mathematical Functions*. Cambridge University Press, New York.

Ziatdinov, R., Yoshida, N., and Kim, T, (2012). Analytic parametric equations of log-aesthetic curves in terms of incomplete gamma functions, Computer Aided Geometric Design, **29** (2), 129-140.

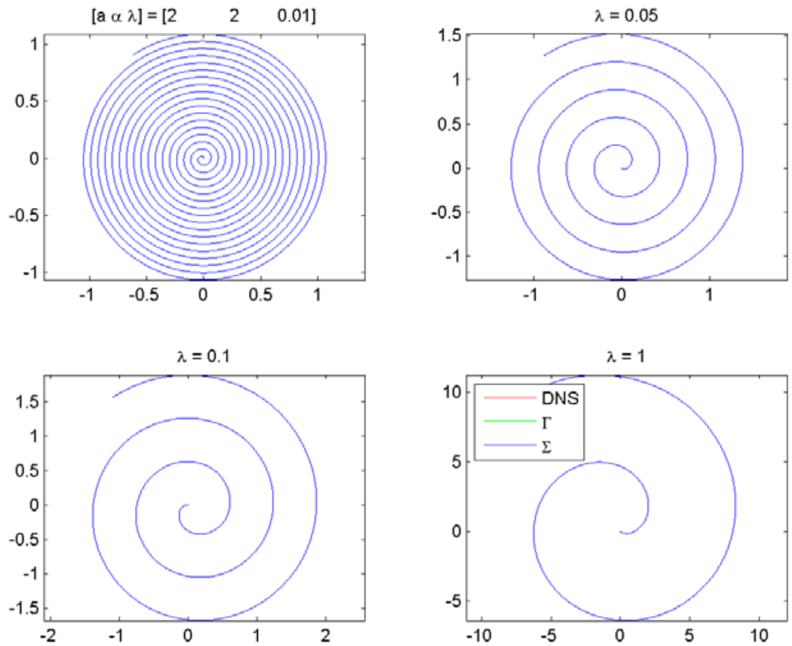

Figure 1: Log-aesthetic curves with *a* = 2 $(\alpha = 2)$. The value of θ is changing from its lower bound to 10 radians.

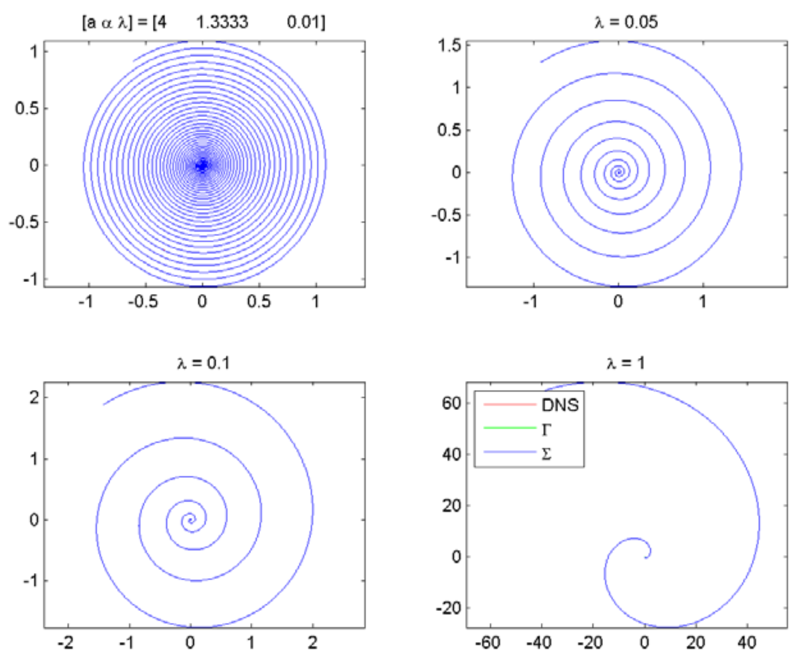

Figure 2: Log-aesthetic curves with *a* = 4 $(\alpha = 4/3)$. The value of θ is changing from its lower bound to 10 radians.

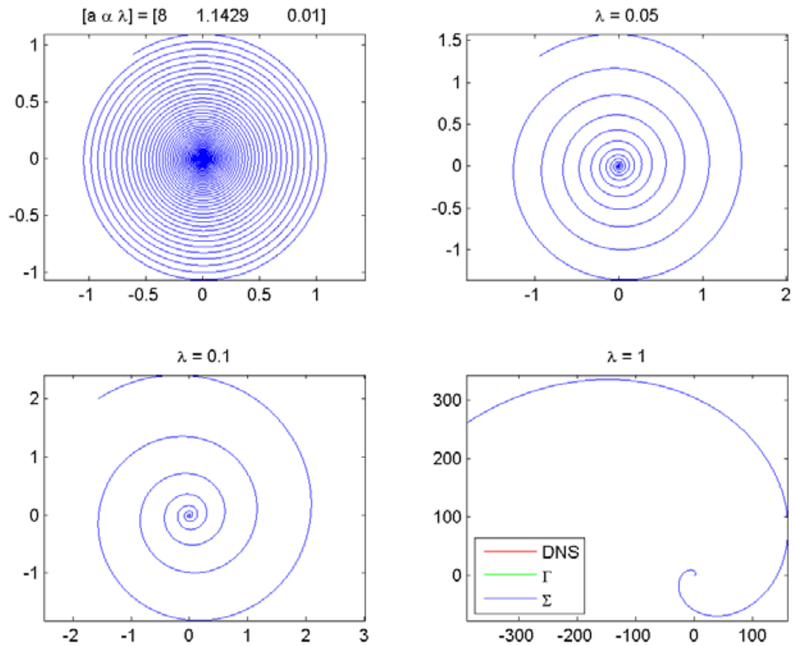

Figure 3: Log-aesthetic curves with *a* = 8 $(\alpha = 8/7)$. The value of θ is changing from its lower bound to 10 radians.

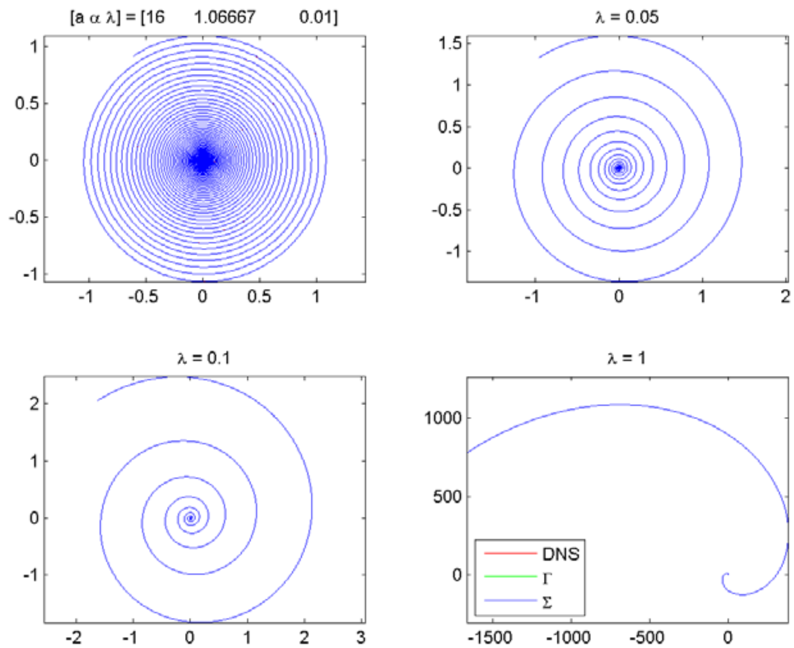

Figure 4: Log-aesthetic curves with *a* = 16 $(\alpha = 16/15)$. The value of θ is changing from its lower bound to 10 radians.

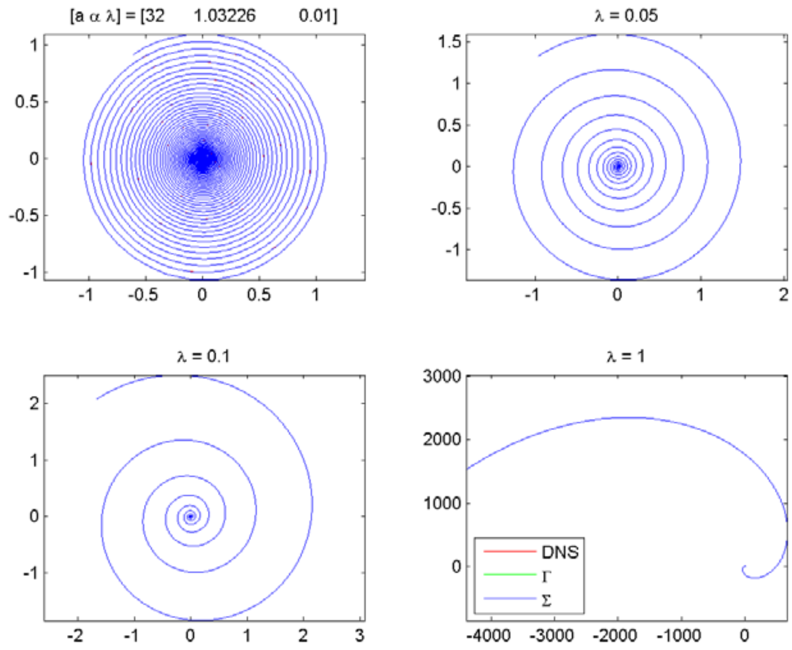

Figure 5: Log-aesthetic curves with *a* = 32 $\left(\alpha = 32/31\right)$. The value of θ is changing from its lower bound to 10 radians.

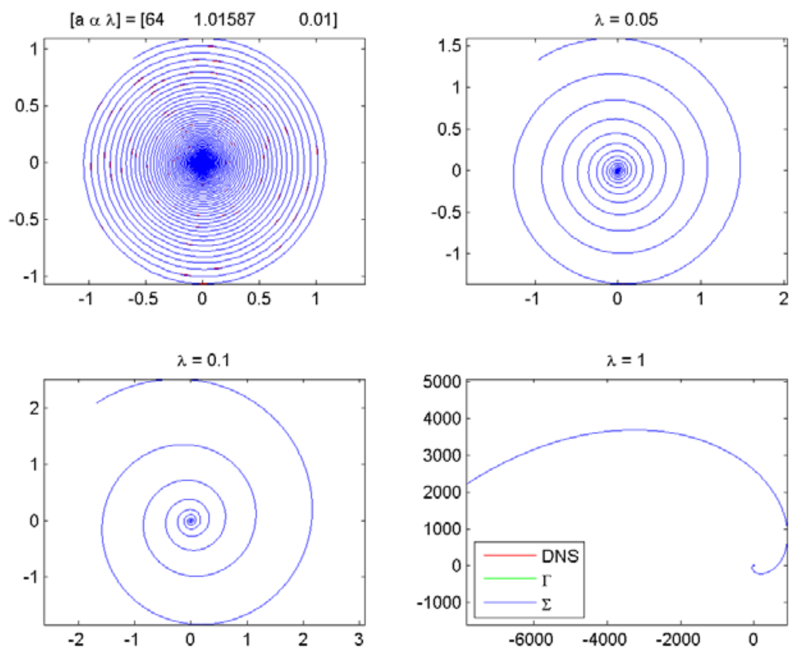

Figure 6: Log-aesthetic curves with *a* = 64 $\left(\alpha = 64/63\right)$. The value of θ is changing from its lower bound to 10 radians.